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BEAM ELIMINATION BASED ON SEQUENTIALLY ESTIMATED A POSTERIORI PROBABILITIES OF WINNING

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ABSTRACT

A robust and adaptive variable length beam selection strategy based on M-ary sequential competition was proposed in [1]. It was enhanced by the elimination of inauspicious beams during the ongoing competition to improve the efficiency and speed of the training in [2]. In this paper, we refine the elimination process by introducing a new elimination mechanism based on estimated winning probability i.e. probability of being the strongest candidate for each beam at each time step. These probabilities are calculated using sequentially estimated a posterirori PDFs of the unknown signal amplitudes after beamforming. This way least promising beams that fail to promise a minimum predefined winning probability can be eliminated from the remaining candidates as early as possible.

Index Terms— Millimeter-wave, Beam Selection, Beam Alignment, Sequential Competition Test, Massive MIMO

1. INTRODUCTION

Communications at higher frequencies like mmWave alleviates the spectrum gridlock while offering larger bandwidth. At the same time, harsher propagation conditions and the sparsity of the channel at higher frequencies motivate the use of beamforming to increase the link budget. This can be achieved in practical multiple antenna systems via electronically controlled beamforming networks which provide a set of beams (denoted as 'codebook') steered into different mainlobe directions [3–12].

An essential part of initial beam training showed schematically in Fig 1, is how to efficiently and reliably select the best beam(s) from a given codebook of current candidate beams in order to achieve the highest possible rate. State of the art beam-alignment techniques such as exhaustive [13], pseudo-exhaustive [14] or tree search with hierarchical codebooks [15] as proposed in IEEE 802.11ad, use a *fixed* length probing sequence to determine the best beam. However, a *fixed* length probing sequence can only be optimally designed for one particular Signal to Noise Ratio (SNR) which is unknown and may vary over at least one order of magnitude in most practical scenarios [16].

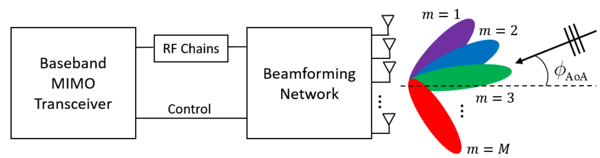

Fig. 1. MIMO transceiver using electronically controllable beamforming network.

A *variable* length approach for beam selection denoted as *sequential competition test* (SCT) was proposed in [1]. It shows adaptivity w.r.t. the SNR operating point and even achieves a shorter average test length compared to the optimally tuned (i.e. one having genie knowledge about SNR) fixed length test at lower SNR values. The idea of beam elimination was first introduced in [2] to increase the efficiency of SCT by shutting down the unpromising beams during the competition. It employed an elimination criterion based on the distance between Generalized Log Likelihood Ratios (GLLRs) of the leading competitor and all other candidate beams to eliminate the beams that are lagging strongly behind.

In this work we present an improved elimination mechanism using estimated *a posteriori* PDFs of the unknown effective signal amplitudes under each beam, acquired via Sequential Linear Minimum Mean Squared Error (SLMMSE) estimator [17] [18]. This algorithm calculates the probability of winning for each beam at each time step. The elimination of any beam occurs as soon as its corresponding estimated winning probability falls bellow a small value. The proposed algorithm shows significant improvement in terms of efficiency and robustness when compared to [2].

2. DATA MODEL

The channel model for mmWave systems [19] contains features like multipath with multiple delays and angle of arrivals (AoA). This leads to the following baseband received sequence $b_m[n]$ after beamforming under each candidate beam pattern $\mathbf{w}_m(\phi_m)$ as

$$b_m[n] = \sum_{p=1}^{P} \underbrace{\rho_p \mathbf{w}_m(\phi_m)\mathbf{a}(\phi_p)}_{A_{m,p}} s[n-\tau_p] + z'_m[n] \,, \quad (1)$$

where n is the sample index, P denotes the number of paths with unknown gains ρ_p, delays τ_p and AoAs ϕ_p. The array response vector is denoted as $\mathbf{a}(\phi_p)$, while ϕ_m stands for the steering direction of beam m. A pseudo-random sequence with $s[n] \in \{\pm 1\}$, variance one and $P\{s[n] = +1\} = P\{s[n] = -1\} = 1/2$ is assumed for training so that $\mathrm{E}[s[n]s[n-\tau]] \simeq \delta[\tau]$ holds for its autocorrelation sequence as n increases. The additive noise $z'[n]$ is modeled as complex zero mean white Gaussian with unknown variance σ^2. The combined effective channel and beamforming gain corresponding to beam m and delay index τ can be written as the unknown parameter $A_{m,\tau}$. The sufficient statistics for the unknown amplitudes $A_{m,\tau}$ at each n are given by the correlated sample mean as

$$\bar{y}_{m,\tau}[n] = \frac{1}{n}\sum_{i=1}^{n} s[i] b_m[i+\tau] = A_{m,\tau} + e_{m,\tau}[n] \,, \quad (2)$$

with $e_{m,\tau}[n]$ being zero mean white Gaussian distributed with $\sigma_e^2 \simeq \sigma^2/n$. For ease of exposition and the fact that essential aspects of the detection problem under consideration are captured by a flat channel model i.e. a single dominant path, we consider the reduced data model by dropping the delay index as

$$y_m[n] = A_m + z_m[n] \,, \quad (3)$$

where $y_m[n]$, A_m and $z_m[n]$ are the correlated observation, the unknown complex signal amplitude and a zero mean complex white Gaussian noise with unknown variance σ^2 under beam m, respectively.

A fixed length test [6] with length N^{fix} decides for beam k over all candidate beams, based on asymptotically efficient signal magnitude estimates, if

$$\mathcal{H}_k : k = \underset{m\in\{1,\ldots,M\}}{\operatorname{argmax}} \{|\bar{y}_m[N^{\text{fix}}]|\} \,, \quad (4)$$

where $|.|$ denotes absolute value operation.

Defining the maximum gain as $|A_{\max}| = \max_{m\in\{1,\ldots,M\}} \{|A_m|\}$ and the vectors of correlated observations $\mathbf{y}_m = y_m[1:n]$, any detector results in the following normalized average loss of the signal magnitude

$$\bar{l} = 1 - \frac{1}{|A_{\max}|}\sum_{m=1}^{M} P\{\mathcal{H}_m|\mathbf{y}_1,\ldots,\mathbf{y}_M\}|A_m| \,, \quad (5)$$

where $P\{k=m|\mathbf{y}_1,\ldots,\mathbf{y}_M\}$ denotes the probability of selecting beam m given the sequences $\mathbf{y}_1,\ldots,\mathbf{y}_M$. Fixed length detectors are prone to SNR change and the achieved performance in terms of $\bar{l}$ can vary greatly. Additionally, if N^{fix} is conservatively set to a high value based on the worst still acceptable operating point, a lot of time spent for detection of the best beam will be wasted, if the channel quality is actually better than expected.

3. SEQUENTIAL COMPETITION TEST AND BEAM ELIMINATION PROBLEM

To solve the problem of deciding as early as possible, which among M unknown amplitude levels is the strongest in a scenario where exact knowledge of the underlying PDFs and in particular the current SNR value is not available, a so-called sequential competition test (SCT) was proposed recently [1]. SCT decomposes the M-ary test into M parallel binary tests by introducing a virtual no-signal (null) hypothesis against which all signals are compared as

$$\begin{array}{ll} \mathcal{H}_m^0 : & y_m[n] = z_m[n] \\ \mathcal{H}_m^1 : & y_m[n] = A_m + z_m[n] \end{array} . \quad (6)$$

The sequence metrics $\gamma_m[n]$ corresponding to each beam m are then calculated based on the GLLR as

$$\gamma_m[n] = n \ln\left(1 + \frac{|\bar{y}_m[n]|^2}{\hat{\sigma}^2[n]}\right) \sim \begin{cases} \chi_1^2, & \text{under } \mathcal{H}_m^0 \\ \chi_1^2(\lambda_m[n]) & \text{under } \mathcal{H}_m^1 \end{cases} \quad (7)$$

where $\bar{y}_m[n] = \sum_{i=1}^{n} y_m[i]/n$ is the sample mean, $\hat{\sigma}^2[n] = \sum_{m=1}^{M}\sum_{i=1}^{n} |y_m[i] - \bar{y}_m[n]|^2/(Mn)$ is the estimated noise variance. $\lambda_m[n] = n|A_m|^2/\sigma^2$ denotes the noncentrality parameter of the chi-squared distribution under $\mathcal{H}_m^1$. Since the PDF of the GLLR under null hypothesis is fully known and independent of n, the probability of false alarm (P_{FA}) corresponding to each binary test can be bounded by choosing the termination threshold as $\gamma_{\text{term}} = [Q^{-1}(P_{\text{FA}}/2)]^2$. Allowing n to grow until a decision criterion is fulfilled, leads to M competing binary variable length tests that check the inequality

$$\gamma_m[n] \underset{\text{undecided}}{\overset{\mathcal{H}_m^1}{\gtrless}} \gamma_{\text{term}} \,. \quad (8)$$

The M-ary test terminates as soon as one of the paths surpasses γ_{term} while the index of this path indicates the selected beam. Otherwise, we continue by taking the next observation. The test length n is now a random variable.

The idea of beam elimination schematically illustrated in Fig. 2 was first proposed in [2] to further improve the performance of SCT in terms of average total number of observations. It has augmented the SCT by an elimination mechanism to drop the beams that fail to keep up with the leading competitor during the competition. In this method, the elimination of a beam takes place when the path metric corresponding to that beam falls bellow a time dependent elimination threshold $\gamma_{\text{elim}}[n]$ which is defined as

$$\gamma_{\text{elim}}[n] = \max_{j\in J}(\gamma_j[n]) - \alpha\gamma_{\text{term}} \,, \quad (9)$$

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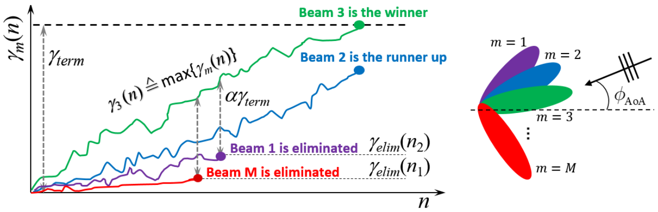

Fig. 2. Illustration of the sequential competition and elimination test in [2].

with a scaling factor $\alpha \in [0,1]$ and the set of current active beams J which is initialized as $J = \{1, 2, \ldots, M\}$ and updated as soon as any beam exits the competition.

In an attempt to evaluate the optimality of the above elimination mechanism, we came up with a new beam elimination technique explained in the next section.

4. BEAM ELIMINATION BASED ON SEQUENTIAL A POSTERIORI PROBABILITIES

The goal of an optimal sequential elimination mechanism is to recognize and eliminate the weakest beams as early as possible. It means that, the weaker a beam looks compared to the strongest candidates, the earlier it should be detected and removed from the set of active candidates. To quantify this, we need to evaluate the probability of being the strongest candidate i.e. winning probability for each beam at each step n.

Considering the unknown signal amplitudes to be random variables, one can find a way to estimate their corresponding a posteriori PDFs $P_n(A_m)$ sequentially after each observation. Let us assume a Gaussian prior PDF [18] for each unknown random variable A_m with the prior mean and variance $\mu_{A_m}[0]$ and $\sigma^2_{A_m}[0]$ before taking any observation. Gaussian priors enable us to use the SLMMSE estimator [17]. This way the estimated a posteriori PDF of A_m denoted as $\hat{P}_n(A_m)$ can be evaluated by its mean (i.e. Bayesian estimate of A_m) $\mu_{A_m}[n] = \hat{A}_m[n] = \mathbb{E}\{A_m|\hat{P}_{n-1}(A_m), y[n]\}$ and the variance $\sigma^2_A[n]$ sequentially after observing the data at each step $n \geqslant 1$ as

$$\hat{A}_m[n] = \hat{A}_m[n-1] + K[n]\left(y_m[n] - \hat{A}_m[n-1]\right)$$
$$\sigma^2_A[n] = (1 - K[n])\,\sigma^2_A[n-1], \tag{10}$$

with weighting factor $K[n]$ also known as Kalman gain

$$K[n] = \frac{\sigma^2_A[n-1]}{\sigma^2_A[n-1] + \hat{\sigma}^2[n]}\,. \tag{11}$$

The SLMMSE treats the estimated a posteriori PDF after last observation i.e. $\hat{P}_{n-1}(A_m)$, as the prior for the next observation $y[n]$. Once the observation is made the a posteriori PDF from last step is updated using the Kalman gain to $\hat{P}_n(A_m)$.

Now that the estimated a posteriori PDFs of the unknown amplitudes are available, one can write the probability of being the strongest candidate i.e. $P^{\text{str}}_m[n] = P\{|A_m| = \max_{j\in J}|A_j|\,|\hat{P}_n(A_j)\}$ for each beam as

$$P^{\text{str}}_m[n] = \int_{-\infty}^{\infty} \hat{P}_n(A_m) \prod_{j\in J\backslash m} P\{|A_m| > |A_j|\}\, \mathrm{d}A_m \tag{12}$$

The evaluation of the above equation is a cumbersome task due to the product of Q-functions within the integral. However, there exist a simple upper bound $P^{\text{str}}_m[n] \leqslant \tilde{P}_m[n]$ as

$$\tilde{P}_m[n] = \int_{-\infty}^{\infty} \hat{P}_n(A_m) P\{|A_m| > |A_{\underset{j\in J\backslash m}{\operatorname{argmax}}|\hat{A}_j[n]|}|\}\, \mathrm{d}A_m, \tag{13}$$

in which the product has been replaced with the dominating term. Eq. 13 is numerically tractable and there even exist an approximate closed form solution [20, 21].

Having a quantified measure in the hand, the elimination rule based on sequential a posteriori can be formulated as

$$\tilde{P}_m[n] \underset{\text{eliminate}}{\overset{\text{keep}}{\gtrless}} P_{\text{elim}}\,, \tag{14}$$

where P_{elim} is the elimination threshold.

The interpretation is that, as soon as a candidate fails to promise a predefined minimum winning probability, it is eliminated from the competition and therefore no further observations under that beam is made. This will increase the efficiency both in terms of time and energy spent for beam selection. Higher energy efficiency is expected since less energy is wasted by the power hungry analog to digital converters (ADCs) at mmWave frequencies to probe unpromising beams. This leads to the following sequential competition and elimination test stated with pseudo code in Algorithm 1.

It is worth noting that, when there remains a sole survivor candidate after eliminating all other candidates, the test will terminate and the survivor will be declared as the winner. This introduces an interplay between termination by reaching the termination threshold (termination by competition) or by being the only survivor (termination by elimination).

5. NUMERICAL EVALUATION

We numerically studied the performance of the SCT augmented with the proposed elimination mechanism in the reference data model described in Eq. (3) with a uniform linear array with 64 antenna elements using the codebook of a Butler matrix with 64 normalized beams. Statistical performance of the proposed test is evaluated via a Monte Carlo simulation with 10^4 iterations. The AoA was distributed uniformly in $[-90°, 90°]$ over the simulation runs while SNR after beamforming was defined as $(1^2/\sigma^2)$[dB]. Similar to $\bar{l}$ in Eq. 5, we evaluate the average relative effective rate $\mathbf{E}[R_{\text{eff}}/R_{\max}]$ after beam selection as a performance indicator. Efficiency of the beam selection with respect to the performance is measured

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Algorithm 1 Sequential Competition and Elimination Test

input: $M, \gamma_{\text{term}}, P_{\text{elim}}, n = N_{\min}, N_{\max}, \gamma_m[1] = 0$
$J = \{1, 2, \ldots, M\}$
$\hat{A}_j[0] = 0, \sigma_A^2[0] = \hat{\sigma}^2[N_{\min}] \rightarrow \hat{A}_j[N_{\min}], \sigma_A^2[N_{\min}]$
while $\max_{j\in J}(\gamma_j[n]) < \gamma_{\text{term}} \wedge |J| \geqslant 2 \wedge n \leqslant N_{\max}$ **do**
  $n = n + 1$
  $\hat{\sigma}^2[n] = \sum_{j\in J}\sum_{i=1}^{n} |y_j[i] - \bar{y}_j[n]|^2/(|J|n)$
  $K[n] = \sigma_A^2[n-1]/(\sigma_A^2[n-1] + \hat{\sigma}^2[n])$
  $\sigma_A^2[n] = (1 - K[n])\sigma_A^2[n-1]$
  for $m \in J$ **do**
    $\gamma_m[n] = n \ln(1 + |\bar{y}_m[n]|^2/\hat{\sigma}^2[n])$
    $\hat{A}_m[n] = \hat{A}_m[n-1] + K[n](y_m[n] - \hat{A}_m[n-1])$
  end for
  for $m \in J$ **do**
    if $\tilde{P}_m[n] < P_{\text{elim}}$ **then**
      $J = J - \{m\}$
    end if
  end for
end while
output: $\operatorname{argmax}_{j\in J}(\gamma_j[n])$

by average total number of observations $\bar{n}_{\text{tot}} = \mathbb{E}[\sum_{m=1}^{M} n_m]$, where n_m indicates the number of observations made for beam m during each iteration. $\mathbf{E}[R_{\text{eff}}/R_{\max}]$ and $\bar{n}_{\text{tot}}$ are evaluated for SNR values in the interval $[-9, 3]$ dB.

For comparison we consider the fixed length detector stated in Eq. 4 with $N^{\text{fix}} \in [75, 150]$, the pure SCT [1] and the SCT augmented with elimination as proposed initially in [2]. Hyper parameters corresponding to different approaches has been chosen in a way that a similar range of performance in terms of average relative effective rate is achieved. As the comparison depicted in Fig. 3 shows, pure SCT is the most adaptive and robust in terms of performance achieving above $\%99$ of the maximum rate on average over the whole range of SNR values. Additionally it is more efficient than the fixed length test at each N_{fix}. SCT augmented with the proposed elimination based on sequentially estimated a posteriori (SaPP) winning probabilities, increases the efficiency of the pure SCT by around two folds while maintaining the adaptivity and robustness of the pure SCT by achieving average effective rate of above $\%98$ over a large range of SNR values. Besides, it outperforms significantly the previously proposed algorithm in [2] in both performance and efficiency while being more robust.

Note that the value of the P_{elim} is heuristically set to 0.1. Choosing a large value for P_{elim} can result in higher probability of wrongful elimination of the true strongest candidate at early stages of the test and therefore hurting the performance of the pure SCT. Due to our numerical investigations the choice of $P_{\text{elim}} \leqslant 0.1$ results in a reasonable and robust performance in large range of SNR values, while reducing greatly the $\bar{n}_{\text{tot}}$ compared to pure SCT.

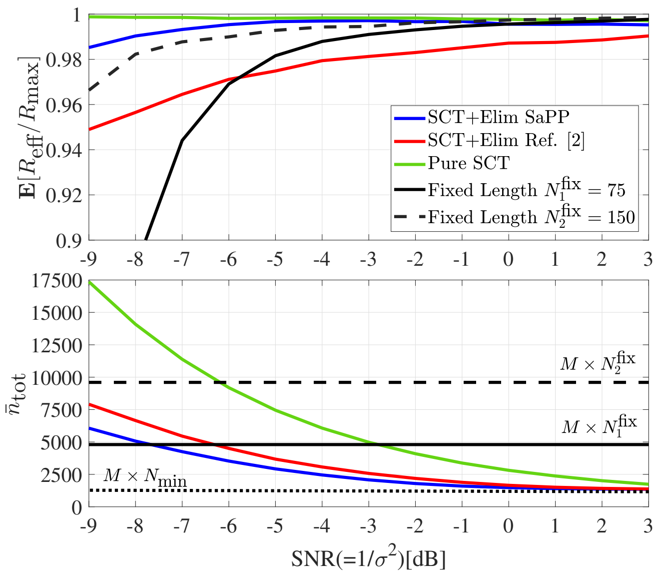

Fig. 3. Performance Comparison using average relative achieved rate $\mathbf{E}[R_{\text{eff}}/R_{\max}]$ and the corresponding average total number of observations $\bar{n}_{\text{tot}}$ for codebook of size $M = 64$ beams and $\gamma_{\text{term}} = 24 (\approx P_{\text{FA}} = 10^{-6})$. SCT+Elim SaPP uses the elimination probability of $P_{\text{elim}} = 0.1$, while SCT+Elim Ref. [2] uses the fraction $\alpha = 0.5$. Initial minimum number of observations was set to $N_{\min} = 20$. Limitted coherence time is not considered in the rate evaluation.

6. CONCLUDING REMARKS

In order to further improve the efficiency of SCT, we have introduced a new beam elimination mechanism based on estimated a posteriori PDFs of the unknown signal amplitudes under different beams, using the sequential linear MMSE estimator. Employing the estimated a posteriori PDFs we calculated the approximate winning probability for each candidate beam at each time step. The beams that fail to promise a predefined minimum winning probability are discarded and no further resources in terms of observations will be allocated to them for probing. As a result, the efficiency both in terms of time and energy spent for beam selection is greatly increased while still enjoying desirable features like adaptivity and robustness of SCT. The same qualitative performance can be expected in more general mmWave multi-path channel.

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