On Reconstructable ASK-Sequences for Receivers employing 1-Bit Quantization and Oversampling

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***Abstract*—Multigigabit communications based on 1-Bit quantization at the receiver benefit from relaxed power consumption as compared to conventional system designs, especially for short range communications. Utilizing oversampling at the receiver compensates partially the loss in terms of achievable rate brought by the coarse quantization by enabling and exploiting the inherent channel memory. This work proposes methods how to construct ASK-sequences which can be perfectly reconstructed at the receiver. Besides this advantage the proposed sequences are also superior in terms of achievable rate as compared to independent uniformly distributed input signals. Simulation based computations of the achievable rate confirm the theory of the model and the benefit of the oversampling approach.**

I. Introduction

When considering communications with extreme bandwidth the analog to digital conversion becomes more important in terms of system design because of its rising power consumption for higher sampling rates and resolution [1]. Especially for short range communications the analog-to-digital conversion becomes the mayor bottleneck. Therefore novel system designs have to be developed which are especially designed for coarse quantization. This work addresses the extreme case of communications employing only a 1-bit quantization at the receiver. The loss in terms of achievable rate is counteracted by sampling with a rate which is an integer multiple of the signaling rate. These extreme sampling rates become feasible when considering for instance the recent BiCMOS technology with a transit frequency of up to 500 GHz. Such an approach is already known from earlier investigations where a bandlimited signal is sampled with a higher rate as compared to Nyquist rate. In this sense, in [2] a marginal benefit in terms of achievable rate from oversampling was reported. Later, significantly higher rates have been proven in [3]. An alternative approach was given by [4] where a bandlimited signal is superposed with a deterministic dither which results in a more accurate signal reconstruction. The pure oversampling approach with 1-bit quantization was further investigated in [5], where the impact of oversampling in the low SNR regime was studied. Besides fundamental studies also practical inspired work can be found in literature. In [6] a conventionally QAM modulated signal benefits from oversampling by exploiting the inherent intersymbol interference in the same way as a statistical dither. The idea was extended in [7] where independent and uniformly distributed 16-QAM signals can be reconstructed perfectly by exploiting an optimized waveform and the structured intersymbol interference in a deterministic way. Nevertheless an optimized waveform might correspond to strong requirements on the analog components which might be not acceptable. This work aims for comparable rates by only considering an appropriate sequence design such that the sequence can be reconstructed at the receiver. Indeed the channel with oversampling inherently is a memory channel where it is known that its capacity can be asymptotically achieved when considering Markov sources [8]. The contribution of the work can be summarized as follows:

- A state machine which generates reconstructable ASK-sequences is proposed.
- Finite state Markov sources have been proposed which approximate the state machine where the current input symbol depends only on a finite number of previous symbols $p(x_k|x_{k-N}^{k-1})$.
- The simulation based computation of the achievable rates confirms the analytically derived rates. This also confirms the suggested unique reconstruction property of the proposed method. In addition the achievable rate based on the constructed sequences is superior as compared to the achievable rate based on independent and uniformly distributed input.

In what follows we use the notation $x_{k-N}^{k} = [x_{k-N}, ..., x_k]^T$ and $x^n = [x_1, ..., x_n]^T$. Stacked vectors, especially oversampling vectors, are denoted as $\boldsymbol{y}^n = [\boldsymbol{y}_1^T, ..., \boldsymbol{y}_n^T]^T$. Probabilities are denoted as $P(\cdot)$ and probability density functions are declared as $p(\cdot)$.

II. System Model

The input symbols are considered to be drawn from a finite set $x_k \in \mathbb{X}$ with a symbol rate $\frac{1}{T_s}$. As illustrated in Fig. 1 the communications system is characterized by the transmit filter $h(t)$, additive white Gaussian noise, the receive filter $g(t)$ and a 1-bit analog-to-digital converter with a sampling rate of $\frac{M}{T_s}$. The waveform corresponding to the concatenation of the filters is given by $v(t) = h(t) * g(t)$. The corresponding output per input symbol is described by a vector with M elements $\boldsymbol{y}_k = [y_{k,0}, ..., y_{k,M-1}]$. According to the waveform the channel is considered to have memory, such that the current output is

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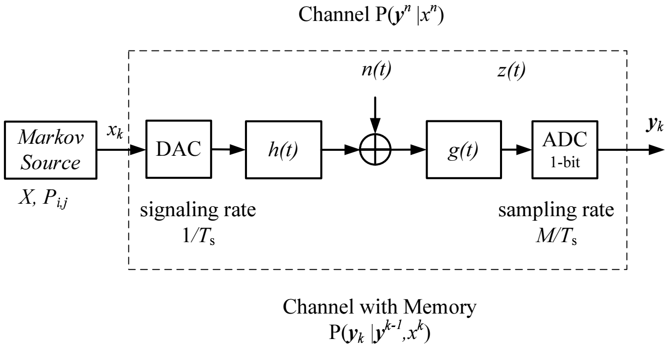

Fig. 1. System Model, 1-Bit quantization and oversampling at the receiver

characterized by the current input and the L previous input symbols (in this work $L = 1$). In what follows the equivalent discrete time representation of the system will be used. The unquantized signal can be denoted as

$$\boldsymbol{z}_k = \boldsymbol{V}\boldsymbol{U}x_{k-L}^{k} + \boldsymbol{G}\boldsymbol{n}_{k-\xi}^{k}, \tag{1}$$

where $\boldsymbol{V}$ is the the joint filter matrix and $\boldsymbol{G}$ is the receive filter matrix. $\boldsymbol{U}$ represents the M-fold upsampling matrix with the dimension $(L+2)M - 1 \times L + 1$ and its elements are given by

$$U_{i,j} = \begin{cases} 1 & \text{for } i = j \cdot M \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The filter matrices containing the discrete time represented reversed filter impulse response function vector $\boldsymbol{v}_\mathrm{r}$ and $\boldsymbol{g}_\mathrm{r}$ have Toeplitz structure as follows

$$\boldsymbol{V} = \begin{pmatrix} \left[\ \boldsymbol{v}_\mathrm{r}^T\ \right] & 0\cdots & 0 \\ 0 & \left[\ \boldsymbol{v}_\mathrm{r}^T\ \right] & 0\cdots 0 \\ & \ddots\ \ddots\ \ddots & \\ 0\cdots & 0 & \left[\ \boldsymbol{v}_\mathrm{r}^T\ \right] \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} \left[\ \boldsymbol{g}_\mathrm{r}^T\ \right] & 0\cdots & 0 & 0 \\ 0 & \left[\ \boldsymbol{g}_\mathrm{r}^T\ \right] & 0\cdots 0 & 0 \\ & \ddots\ \ddots\ \ddots & & \\ 0\cdots & 0 & \left[\ \boldsymbol{g}_\mathrm{r}^T\ \right] & 0 \end{pmatrix}, \tag{3}$$

where $\boldsymbol{V}$ has the dimension $M \times M(L+2) - 1$ and $\boldsymbol{G}$ has the dimension $M \times M(\xi + 1)$. ξT_s is the length of the impulse response of the receive filter. $\boldsymbol{n}_k = [n_{k,0}, ..., n_{k,M-1}]$ are independent and identically distributed noise realizations with variance σ_n^2. The receive filter has a unit energy property such that the filtered noise also has variance σ_n^2. Finally the received signal is converted into a vector of binary symbols

$$\boldsymbol{y}_k = Q\{\boldsymbol{z}_k\}, \tag{4}$$

where $Q(z \geq 0) = 1$ respectively $Q(z < 0) = -1$ denotes the quantization. A special case for demonstration purpose is considered with $M = 3$, $L = 1$, $\xi = 1$. The waveform respectively receive filter is given by

$$\boldsymbol{v} = \left[\frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0\right]^T, \quad \boldsymbol{g} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T. \tag{5}$$

The input alphabet is taken from a regular 4-ASK modulation, where $E\{|x_k|^2\} = 1$. The sampling points in time domain are chosen such that the maximum value of the waveform is included and two samples between the symbols. We believe that the proposed methods which will be introduced in the next sections can be extended in various directions with reasonable effort. As it will be discussed in the next section, the symbol transitions become essential for sequence reconstruction purpose.

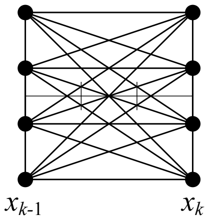

Fig. 3. i.u.d. symbol transitions

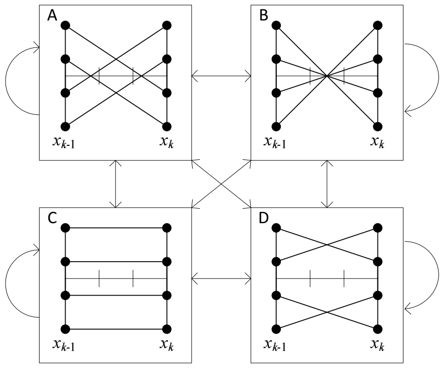

Fig. 4. i.u.d. sequence as a realization of the state machine

III. Input Sequences Design

A. Independent and Uniformly Distributed Input

When considering the input entropy rate it turns out that independent and uniformly distributed (i.u.d.) symbols corresponding to 2 bits per symbol are the optimal choice. The possible signal transitions and the waveforms are illustrated in Fig. 3, where two bars in the middle of two sequential symbols indicate the additional sampling points in time domain. With respect to the constrained receiver design the signal reception suffers from the coarse quantization and a number of sequences cannot be reconstructed reliably. In order to gain understanding of the reconstruction problem it useful to distinguish between classes of symbol transitions. In the present example 3 respectively 4 classes have been identified and they are called transition states. Each individual sequence can be represented as realization of a state machine drawn in Fig. 4. According to the binary observation at the receiver with 3-fold oversampling the transition states named A, B, C and D have special properties in terms of sequence reconstruction:

- A: x_{k-1} and x_k can be directly reconstructed based on the current channel output (“Decision”)
- B: x_{k-1} and x_k can be reconstructed when x_{k-1} or x_k is known at the receiver (“Forward”)
- C: possible ambiguity with state D (“Ambiguity1”)
- D: possible ambiguity with state C (“Ambiguity2”)

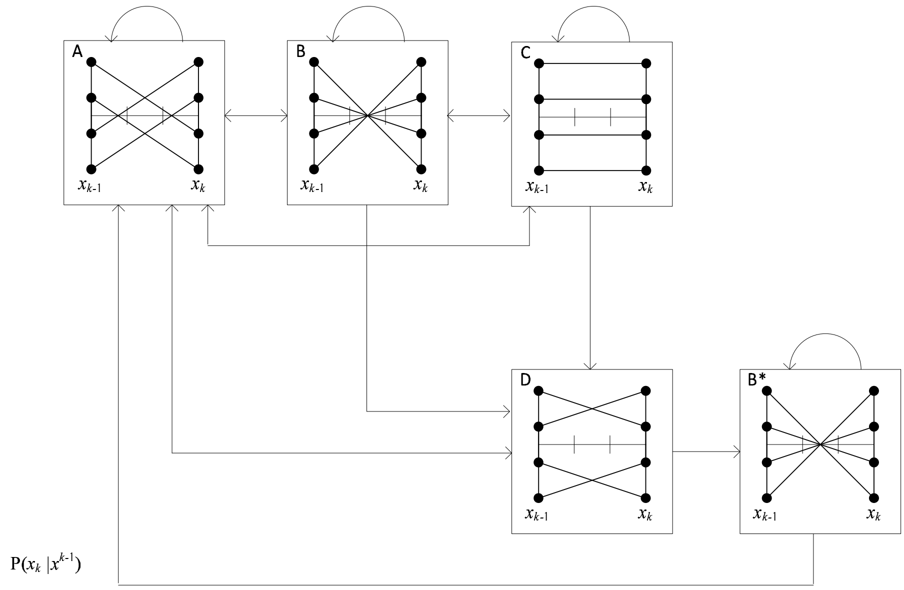

Fig. 2. Reconstructable sequences as a realization out of the state machine, with $p(x_k|x^{k-1})$, mind the directivity of connections

Due to the fact that not all sequences can be distinguished at the receiver the achievable rate will be below the source entropy rate of 2 bits per symbol. It can be expected that the achievable rate will be below the input entropy rate. In the next subsections the transition states are utilized for the design of structured sequences that can be reconstructed.

B. Reconstructable Sequences

The sequence reconstruction problem occurs when the ambiguity states are passed in a random way. One approach to cope with the problem is to apply a special rule for one of the ambiguity states "C" or "D". In the proposed approach the state "D" has been selected. The special rule implies that after passing the "D" transition state and until passing the next "A" transition state only "B" transition states are allowed to occur. "B" transition states which occur subsequently to a "D" are named "B*". The "A" state ("Decision") terminates a so far ambiguous phrase. The corresponding modified state machine is drawn in Fig. 2.

The adjacency matrix for the modified state machine is given by

$$\boldsymbol{D} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Shannon showed in [9] that the maximum entropy rate of the proposed state machine is given by,

$$\begin{aligned} H_{\max} &= \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{i,j} [\boldsymbol{D}^n]_{i,j} \\ &= \log_2(\lambda) \\ &= 1.7716 \text{ Bits}, \end{aligned} \tag{7}$$

where λ is the largest real valued eigenvalue of $\boldsymbol{D}$. Furthermore following [9] the corresponding transition probabilities are given by

$$P_{i,j} = \frac{b_j}{b_i} \cdot \frac{D_{i,j}}{\lambda}, \tag{8}$$

where b_i and b_j are entries of the right-sided eigenvector corresponding to the eigenvalue λ.

Each transition probability is determined by previous symbols realizations $p(x_k|x^{k-1})$. Nevertheless the number of previous symbols is not limited. As a result this source model cannot be described by a state model where each state equals to a number of previous symbols, which might be favorable. An alternative model providing this property is given in the next section.

C. Reconstructable Finite State Markov Source

In order to obtain a source model where each output depends on a finite number of previous symbols

$$P(x_k|x^{k-1}) = P(x_k|x_{k-N}^{k-1}) \tag{9}$$

further modifications can be included successively. For specific N the following requirements need to be fulfilled:

- $N = 1$ avoiding state "D"
- $N = 2$ state "A" is directly enforced when "D" occurs: "DA"
- $N = 3$ state "A" is enforced when "D" occurs: "DA" or "DB*A"
- $N = 4$ state "A" is enforced when "D" occurs: "DA" or "DB*A" or "DB*B**A"

The corresponding state machines are illustrated in Fig. 5. Analogously to previous section the adjacency matrix can be

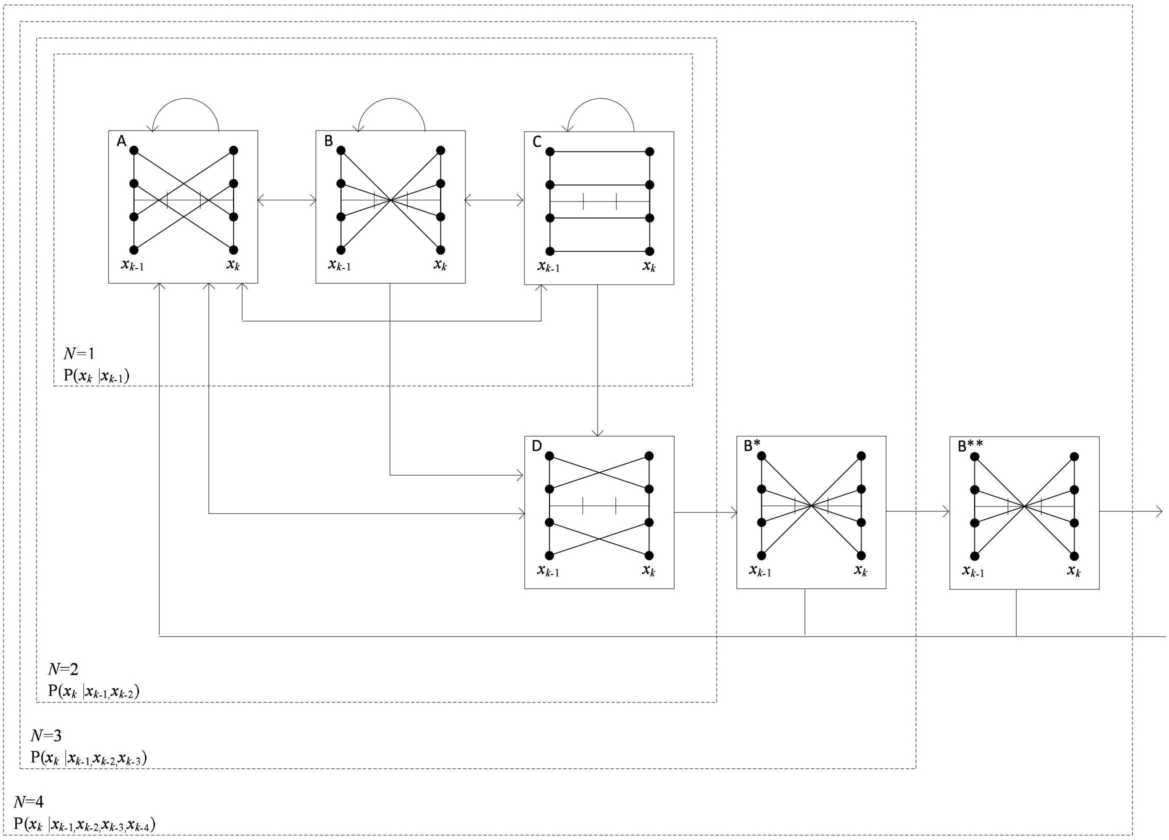

Fig. 5. reconstructable sequences with $p(x_k|x_{k-N}^{k-1})$ property, mind the directivity of connections

TABLE I
SOURCE ENTROPYS RATES OF RECONSTRUCTABLE SEQUENCES AND INDEPENDENT UNIFORMLY DISTRIBUTED SYMBOLS

sequence property	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = \infty$	i.u.d.
$H(X)$ / [bits/symb]	1.585	1.7237	1.7583	1.7678	1.7716	2

found, e.g. for $N = 1$ and $N = 2$

$$\boldsymbol{D}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{D}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

Indeed the maximum entropy rate of the modified source comes close to the original rate (7) already for $N = 4$ as presented in Table I. Therefore we do not consider larger values of N. When translating the proposed state machine into a process where the state is given by the number of previous symbols $s_{k-1} = x_{k-N}^{k-1}$ an equivalent adjacency matrix can be found $\tilde{\boldsymbol{D}}$. Applying (8) to adjacency matrix $\mathit{leadstothetransitionprobabilities}\mathrm{P}_{i,j} = P(s_k|s_{k-1}) = P(x_k|x_{k-N}^{k-1})$ corresponding to maximum entropy.

IV. ACHIEVABLE RATE

A. Simulation based Computation

The achievable rate of a memory channel can be computed by the methods suggested in [10] and [11] employing the Shannon-McMillan-Breiman theorem

$$\lim_{n\to\infty} \frac{1}{n}\hat{I}(X^n;\boldsymbol{Y}^n) = \\ \frac{1}{n}(-\log P(\boldsymbol{y}^n) + \log P(\boldsymbol{y}^n|x^n)P(x^n) - \log P(x^n)). \tag{11}$$

When introducing an auxiliary channel model $W(\cdot)$ with the properties $W(\boldsymbol{y}^n) \approx P(\boldsymbol{y}^n)$ and $W(\boldsymbol{y}^n|x^n) \approx P(\boldsymbol{y}^n|x^n)$ the auxiliary channel lower bound holds. It can be computed with

$$\lim_{n\to\infty} \frac{1}{n}I(X^n;\boldsymbol{Y}^n) \geq \\ \frac{1}{n}(-\log W(\boldsymbol{y}^n) + \log W(\boldsymbol{y}^n|x^n)P(x^n) - \log P(x^n)), \tag{12}$$

where $W(\cdot) > 0$ holds whenever $P(\cdot) > 0$. In this study the auxiliary channel is given by

$$P(\boldsymbol{y}_k|\boldsymbol{y}^{k-1}, x^k) \approx P(\boldsymbol{y}_k|x_{k-1}^k), \tag{13}$$

which becomes for rising SNR values an asymptotically close approximation. This channel assumption will be used to compute the probabilities with the forward recursion of the BCJR algorithm

$$P(\boldsymbol{y}^k) \geq W(\boldsymbol{y}^k) = \sum_{s_k} W(\boldsymbol{y}^k, s_k) = \sum_{s_k} \mu_k(s_k),$$
$$P(\boldsymbol{y}^k|x^n) \geq W(\boldsymbol{y}^k|x^n) = P(\boldsymbol{y}_k|x_{k-N}^k)\tilde{\mu}_{k-1},$$
$$P(x^k) = P(x_k|x_{k-N}^{k-1})P(x^{k-1}), \tag{14}$$

where $s_k = x_{k-N+1}^k$. Each recursion implies

$$\mu_k(s_k) = \sum_{x_{k-N}^{k-1}} P(\boldsymbol{y}_k|x_{k-N}^k)P(x_k|x_{k-N}^{k-1})\mu_{k-1}(s_{k-1}) \tag{15}$$

$$\tilde{\mu}_k = P(\boldsymbol{y}_k|x_{k-N}^k)\tilde{\mu}_{k-1}, \tag{16}$$

where $P(\boldsymbol{y}_k|x_{k-N}^k) = P(\boldsymbol{y}_k|x_{k-L}^k)$ for $L < N$.

B. Transition Probabilities

The probability density function of the unquantized received signal is given by

$$p(\boldsymbol{z}_k|x_{k-L}^k) = \frac{1}{\pi^M |\boldsymbol{R}|} \exp\left(-(\boldsymbol{z}_k - \boldsymbol{\mu}_x)^H \boldsymbol{R}^{-1}(\boldsymbol{z}_k - \boldsymbol{\mu}_x)\right), \tag{17}$$

where the mean is defined as $\boldsymbol{\mu}_x = \boldsymbol{VU}x_{k-L}^k$ and the covariance $\boldsymbol{R} = E\{\boldsymbol{G}\boldsymbol{n}_{k-\xi}^k(\boldsymbol{n}_{k-\xi}^k)^H\boldsymbol{G}^H\}$. In the considered case the covariance is given by

$$\boldsymbol{R} = \sigma_n^2 \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}. \tag{18}$$

V. Numerical Results

The simulation based computation of the achievable rate is carried out based on sequences with a length of $n = 10^5$ symbols. The SNR is defined as the ratio between signal energy and noise power density and given by $\frac{E\{|x|^2\}}{\sigma_n^2}$. The achievable rates illustrated in Fig. 6 indicate a strong benefit of oversampling in terms of achievable rate. As expected the independent and uniformly distributed input symbols can only partially recovered at the receiver as the achievable rate is significantly lower than 2 bpcu. The reconstructable sequences show a saturation in terms of rate at their input entropy rates. This can be interpreted as a numerical confirmation of their reconstruction property. The Markov source input sequences are superior to i.u.d input when considering a source memory of more or equal to 2 previous symbols ($N \geq 2$). The results are compared with the achievable rate considering unquantized received signals and with the achievable rate considering the receiver with 1-bit quantization sampling at symbol rate.

VI. Conclusion

The proposed method allows reliable communications when considering a constrained receiver with a 1-Bit quantization and oversampling with significantly more than 1 bit per channel use. A structured way for reconstructable sequence generation is proposed. The proposed approach can be asymptotically closely approximated by using a finite state Markov source model. Besides the advantage regarding generation of distinguishable ASK sequences the proposed approach is also superior in terms of achievable rate as compared to i.u.d. input symbols.

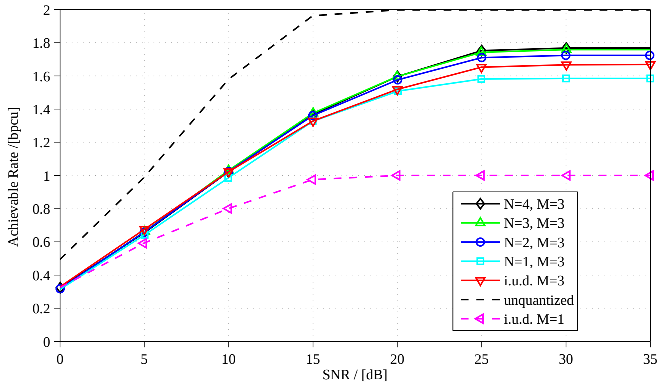

Fig. 6. Achievable rate versus SNR, 4-ASK sequences, M-fold oversampling, Markov source model $P(x_k|x_{k-N}^{k-1})$

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