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Achievable Rates Of Half-Duplex Relay Networks

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***Abstract*— This paper discusses two coding approaches for a network of half-duplex relay terminals and presents achievable rates for the discrete memoryless relay channel. Both proposals gain on an implicit (in case of compress-and-forward) or explicit (in case of decode-and-forward) inter-relay cooperation.**

I. Introduction and motivation

In [1] Cover and El Gamal proposed two fundamental coding strategies for the three-terminal relay channel: *decode-and-forward* and *compress-and-forward*. More recently, Kramer *et al.* presented in their comprehensive work [2] different protocols for the T-terminal relay network. Both papers concentrated on full-duplex relay terminals which are hard to implement cost-efficiently due to practical constraints. Laneman *et al.* investigated this problem in [3] and proposed different *cooperative relaying* protocols for *half-duplex relay terminals*. We apply the ideas of compress-and-forward and decode-and-forward to a network of half-duplex relay terminals to exploit an additional inter-relay cooperation. The advantages of this proposal are that a) it supports a continuous source-destination transmission, b) it ensures that the *complete* source message is retransmitted by the relay nodes, and c) it offers an additional inter-relay cooperation. The paper is structured as follows: Section II presents the underlying relay network model and nomenclature. We proceed in Section III with the description of two proposals for the half-duplex relay network and present achievable rates. Section IV concludes the paper with an outlook.

II. Relay network model and nomenclature

In the following we will use $\underline{x}$ to denote vectors, $\mathcal{X}$ to denote ordered sets, $\|\mathcal{X}\|$ to denote the cardinality of a set and $[b; b+k]$ to denote a set of numbers $(b, \cdots, b+k)$ with $[b; b+k] = \emptyset$ for $k < 0$. We further define the index r over the set $[1; N]$ and define addition using the modulo, i. e., $r+k := \mod(r+k-1, N)+1$. Let $\pi(\mathcal{R})$ be the set of all permutations of a set $\mathcal{R}$ and $\pi_j(r)$ the r-th element in $\pi_j \in \pi(\mathcal{R})$.

We consider in this paper a network of $N+2$ nodes: the set of N relay nodes $t \in \mathcal{R} := [1; N]$, the source node $s = N+1$ and the destination node $d = N+2$. The relay channel is defined over all possible channel inputs $(x_1, \cdots, x_N, x_s) \in \mathcal{X}_1 \times \cdots \mathcal{X}_N \times \mathcal{X}_s$ and channel outputs $(y_1, \cdots, y_N, y_d) \in \mathcal{Y}_1 \times \cdots \mathcal{Y}_N \times \mathcal{Y}_d$ with $\mathcal{X}_i$ and $\mathcal{Y}_j$ denoting the input and output alphabets, respectively. We further use $\underline{y}_{\mathcal{R}}$ to symbolize the vector of all y_t with $t \in \mathcal{R}$. Using this notation the discrete memoryless relay channel is defined by the joint pdf $p\left(\underline{y}_{\mathcal{R}}, y_d | x_s, \underline{x}_{\mathcal{R}}\right)$. To increase the readability we will use in the following the index r as a short-cut for the relay node $t = \pi_j(r)$ when this does not create any confusion.

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III. Protocols for half-duplex relay networks

This section presents achievable rates of two half-duplex relay network proposals.

A. Compress-and-Forward

The first approach generalizes the compress-and-forward protocol presented in [1, Theorem 6]. Using Wyner-Ziv coding [4] the destination exploits Y_d as side information to decode a quantized version $\hat{Y}_r$ of Y_r. Using its own observation Y_d and the quantized relay output $\hat{Y}_r$ the destination decodes X_s.

We generalize this now to a network of half-duplex relays as follows: choose an ordered set $\pi_j \in \pi(\mathcal{R})$ and let the relays transmit in a circular manner, i. e., in the order given by π_j. At a particular time only one relay is transmitting, all others are receiving the source and relay signal. Hence, each relay can observe $N-1$ consecutive channel outputs $y_r(b-1), \cdots, y_r(b-N+1)$ for which it selects the quantized versions $\hat{y}_{r,k}(u_{r,b-k}|x_{r-k}(s_{b-k}))$, $k \in [1; N-1]$, according to a distortion measure $d(y_r, \hat{y}_r)$ where $\hat{y}_{r,k}(u_{r,b-k})$ denotes the quantized version of the channel output $y_r(b-k)$. By the Wyner-Ziv coding approach these estimates determine the sent relay signals $x_r(s_b)$. For the decoding of the estimates the destination exploits $y_d(b-k)$, $k \in [1; N-1]$, as well as the estimates decoded after blocks $[b-1; b-N+2]$ (offering an implicit inter-relay cooperation). Using $\hat{y}_{r-k+1,N-k}(u_{r-k+1,b-N+1}|x_{r-N+1}(s_{b-N+1}))$, $k \in [1; N-1]$, and the own observation $y_d(b-N+1)$ the destination decodes the source index w_{b-N+1} after block b.

An exemplary outline of this coding approach is given in Table I (the table only shows the abbreviated version $\hat{y}_{r,k}(u_{r,b-k})$ for reasons of brevity). Consider the source block $x_s(w_4)$ in this table. Relay nodes $t = 2$ and $t = 3$ are transmitting the index determined by the estimates $\hat{y}_{2,1}(u_{2,4}|x_1(s_4))$ and $\hat{y}_{3,2}(u_{3,4}|x_1(s_4))$. The destination uses $y_d(4)$ to decode $\hat{y}_{2,1}(u_{2,4})$ and decodes $\hat{y}_{3,2}(u_{3,4})$ using $y_d(4)$ and $\hat{y}_{2,1}(u_{2,4})$. Using both estimates and the own channel output the destination decodes $x_s(w_4)$ at the end of block $b = 6$. As outlined in [2] we achieve with this approach a multi-antenna reception behavior. From the previous description we can state the following theorem on the achievable rate.

TABLE I
OUTLINE OF THE COMPRESS-AND-FORWARD BASED CODING SCHEME WITH $B = 5$ SOURCE BLOCKS AND $\pi_j = \{1, 2, 3\}$.

b	s	$t=1$	$t=2$	$t=3$
1	$x_s(w_1)$	$\emptyset, \emptyset$		
2	$x_s(w_2)$		$\hat{y}_{2,1}(u_{2,1})$, $\emptyset$	
3	$x_s(w_3)$			$\hat{y}_{3,1}(u_{3,2})$, $\hat{y}_{3,2}(u_{3,1})$
4	$x_s(w_4)$	$\hat{y}_{1,1}(u_{1,3})$, $\hat{y}_{1,2}(u_{1,2})$		
5	$x_s(w_5)$		$\hat{y}_{2,1}(u_{2,4})$, $\hat{y}_{2,2}(u_{2,3})$	
6	$\emptyset$			$\hat{y}_{3,1}(u_{3,5})$, $\hat{y}_{3,2}(u_{3,4})$
7	$\emptyset$	$\emptyset$, $\hat{y}_{1,2}(u_{1,5})$		

Theorem 1: The achievable rate of the previously described compress-and-forward approach is given by

$$R \leq \max_{\pi_j} \min_{r} \sup I\left(X_s; Y_d, \left\{k \in [1; N-1] : \hat{Y}_{(r-k+1),(N-k)}\right\} | X_{r-N+1}\right) \tag{1}$$

with the side conditions ($r \in [1; N], k, l \in [1; N-1]$)

$$\sum_k R_{r,k} = R_r < \min\left(I(X_r; Y_d), I(X_r; Y_{r+l})\right) \tag{2}$$

$$R_{(r+k),k} > I\left(\hat{Y}_{(r+k),k}; Y_{r+k} | X_r, Y_d, \left\{l \in [2; k] : \hat{Y}_{(r+k-l+1),(k-l+1)}\right\}\right), \tag{3}$$

where (2) is due to the inter-relay and relay-destination communication and (3) is implied by the Wyner-Ziv coding. The supremum is taken over the joint pdf (with $r' \in [1; N] \setminus \{r\}$)

$$p\left(x_s, x_r, \left\{r' \in [1; N] \setminus \{r\} : \hat{y}_{r',(r'-r)}\right\}, y_{\mathcal{R}\setminus\{\pi_j(r)\}}, y_d\right) = p(x_s)p(x_r)p(y_d, \underline{y}_{\mathcal{R}\setminus\{\pi_j(r)\}} | x_s, x_r) \prod_{r'} p(\hat{y}_{r',(r'-r)} | y_{r'}, x_r),$$

which depends on the current $\pi_j \in \pi(\mathcal{R})$ and transmitting r.

Proof: The proof is given in Appendix I. ■

If all $\hat{y}_{r,k}$ are quantized with the same distortion D_r it follows that $R_{r,1} \geq \cdots \geq R_{r,(N-1)}$. For two relays at almost the same position (1) simplifies asymptotically to [1, Theorem 6].

B. Decode-and-Forward

The second approach applies the decode-and-forward strategy presented in [1, Theorem 1]. Using the random binning argument a relay node assigns a partition index to the decoded source message and transmits this index. The source then decodes this index and uses it to decode X_s.

We apply now the idea of decode-and-forward in a similar way as above in the compress-and-forward case. Let relay r be transmitting in block b, it decodes at the end of block $b-1$ (not necessarily all) the source messages $x_s(b-1), \cdots, x_s(b-N+1)$ and transmits the partition indices with $x_r(s_b)$. This offers the chance that each relay can gain on the information sent by the other relays to improve its own decoding. Furthermore, we allow our protocol that if $R_{r,k} = 0$ relay r is not decoding $x_s(b-k)$ since it might happen that one relay relies on the support of other relays to decode the source message. Therefore, not necessarily all relay nodes decode all source messages but we require all relays to decode other relay transmissions. Again the order in which the relays are transmitting is of essential matter, therefore we will again maximize over $\pi(\mathcal{R})$ and let the relays transmit in a circular manner, i. e., relay $r' = r + k$ transmits in block $b' = b + k$.

Theorem 2: The achievable rate of the decode-and-forward based approach is given by

$$R < \max_{\pi_j} \min_{r} \sup \min(R_{s,d}, R_{s,r}) \tag{4}$$

$$R_{s,d} = I(X_s; Y_d | X_{r-N+1}) + \sum_{k=1}^{N-1} R_{(r-k+1),(N-k)} \tag{5}$$

$$R_{s,r} = \min_{\substack{k\in[1;N-1]\\ R_{r,k}>0}} I(X_s; Y_r | X_{r-k}) + \sum_{l=1}^{k-1} R_{r-l,k-l} \tag{6}$$

with the side condition

$$R_r < \min\left(I(X_r; Y_d), I(X_r; Y_{r+k})\right), \tag{7}$$

for $r \in [1; N]$, $k \in [1; N-1]$ and the joint pdf

$$p\left(x_s, x_r, \underline{y}_{\mathcal{R}\setminus\{\pi_j(r)\}}, y_d\right) = p(x_r)p(x_s|x_r)p\left(\underline{y}_{\mathcal{R}\setminus\{\pi_j(r)\}}, y_d | x_s, x_r\right) \tag{8}$$

which depends on the currently transmitting r. The supremum in (4) is over all joint pdf $p\left(x_s, x_r, \underline{y}_{\mathcal{R}\setminus\{\pi_j(r)\}}, y_d\right)$. Furthermore, we need to take the minimum over all relays $r \in [1; N]$ and the maximum over all possible orders $\pi_j \in \pi(\mathcal{R})$.

Proof: The proof is given in Appendix II. ■

If we apply (4) to two relays located at almost the same position it simplifies asymptotically to [1, Theorem 1]. Further consider the case that all $R_{r,k} > 0$, (6) simplifies to $R_{s,r} < \min_{r,k} I\left(X_s; Y_r | X_{r-k}\right)$, a rather tight condition.

IV. FURTHER WORK

Our further work includes the presentation of the achievable rate of a mixed strategy where each relay node can use the other relay quantizations to decode the source messages. Furthermore, we will show results for a network with more than one transmitting relay as well as results for wireless models, e. g., Gaussian relay channel and fading channels.

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APPENDIX I
PROOF OF THEOREM 1

Proof: We give here an outline of the proof for the achievability of the rate given in (1). The proof is intentionally created in the same way as [1, Theorem 6] to allow an easy understanding of the proposed protocol: at first we describe a random coding scheme used in our proof. Then we describe the encoding and decoding procedure to achieve the described rate. The proof relies on the application of the Markov lemma [5, Lemma 14.8.1] which requires strong typicality [5, Ch. 13.6]. We will use in the following the notation defined in Section II and used in [5].

a) Random coding:

1) The source creates 2^{nR} i.i.d. n-length sequences $x_s(w)$ each with $p(x_s) = \prod_{j=1}^{n} p(x_{sj})$, $w \in \left[1, \ldots, 2^{nR}\right]$.
2) Each relay r creates a codebook consisting of 2^{nR_r} i.i.d. n-length sequences $x_r(s_r)$, $s_r \in \left[1; 2^{nR_r}\right]$, each with probability $p(x_r) = \prod_{j=1}^{n} p(x_{rj})$. Further let s_r be the vector of the indices $\left(s_{r,1}, \cdots, s_{r,(N-1)}\right)$, $s_{r,k} \in \left[1; 2^{nR_{r,k}}\right]$, such that $R_r = \sum_{k=1}^{N-1} R_{r,k}$.
3) Create, for each $x_{r-k}(s_r)$ with $r \in [1;N]$, $k \in [1;N-1]$, $s_r \in \left[1; 2^{nR_{r-k}}\right]$, $2^{n\hat{R}_{r,k}}$ i.i.d. n-length sequences $\hat{y}_{r,k}(u^{(k)}|x_{r-k}(s_r))$, each with probability $p(\hat{y}_{r,k}|x_{r-k}(s_r)) = \prod_{j=1}^{n} p(\hat{y}_{r,kj}|x_{(r-k)j}(s_r))$, $u^{(k)} \in \left[1; 2^{\hat{R}_{r,k}}\right]$.
4) We further introduce a random partitioning at each relay r with $N-1$ mappings such that $\hat{y}_{r,k}(u^{(k)}|x_{r-k}(s))$ is randomly mapped independently into $2^{nR_{r,k}}$ cells $S_{r,k}(s_k)$, $s_k \in \left[1; 2^{nR_{r,k}}\right]$, $r \in [1;N]$, $k \in [1;N-1]$, according to a uniform distribution such that each $u^{(k)}$ is uniquely assigned to a bin $S_{r,k}(s_k)$.

b) Encoding: Lets assume that the relay r, transmitting in block b, successfully decoded $x_{r-k}(s_{b-k})$, and it created for each of the last $N-1$ observations the tuple $(y_r(b-k), \hat{y}_{r,k}(u_{r,b-k}|x_{r-k}(s_{b-k})), x_{r-k}(s_{b-k})) \in A_\epsilon^{*(n)}, k \in [1;N-1]$. If $u_{r,b-k} \in S_{r,k}(s_{b,k})$ it transmits $x_r(s_b)$ with $s_b = \left(s_{b,1}, \ldots, s_{b,(N-1)}\right)$, $s_{b,k} \in \left[1; 2^{nR_{r,k}}\right]$. Concurrently the source sends $x_s(w_b)$. We assume that the previous $N-1$ steps were error free. Furthermore, this presentation is done for a certain r but can be similarly done for all $r \in [1;N]$.

c) Decoding: At the end of block b the following decoding procedure is done (at the end of block $b > N-1$ the source index w_{b-N+1} is decoded):

1) The destination at first decodes $x_r(s_b)$. This is done by searching for a uniquely typical $x_r(s_b)$ with $y_d(b)$ which is possible with arbitrarily low probability of error iff $R_r < I(X_r; Y_d)$ and n sufficiently large (resulting from the channel coding theorem).
2) In step 2 the destination creates the sets
$$\begin{aligned}\mathcal{L}_k\left(y_d(b-k)\right) := \{&\tilde{u}_{r,b-k} : (\{l \in [1;k] : \\ &\hat{y}_{(r-l+1),(k-l+1)}(\tilde{u}_{(r-l+1),(b-k)}|x_{r-k}(s_{b-k}))\}, \cdots \\ &x_{r-k}(s_{b-k}), y_d(b-k)) \in A_\epsilon^{*(n)}\}\end{aligned}$$
for $k \in [1;N-1]$. The decoding after block $b-1$ already ensured for $k \in [2;N-1]$
$$\begin{aligned}&(x_{r-k}(s_{b-k}), y_d(b-k), \{l \in [2;k] : \cdots \\ &\hat{y}_{(r-l+1),(k-l+1)}(u_{(r-l+1),(b-k)}|x_{r-k}(s_{b-k}))\}) \in A_\epsilon^{*(n)}\end{aligned}$$
(which are all known to the destination at the end of block $b-1$). Afterwards it chooses for $k \in [1;N-1]$ the estimates $\hat{y}_r\left(\tilde{u}_{r,b-k}|x_{r-k}\left(s_{b-k}\right)\right)$ such that
$$\exists \tilde{u}_{r,b-k} : \tilde{u}_{r,b-k} \in S_{r,k}(s_{b,k}) \cap \mathcal{L}_k\left(y_d(b-k)\right)$$
which succeeds for $k \in [1;N-1]$, i.e., $\tilde{u}_{r,b-k} = u_{r,b-k}$, with arbitrarily low probability of error iff
$$\begin{aligned}\hat{R}_{r,k} < &I\Big(\hat{Y}_{r,k}; Y_d, \{l \in [2;k] : \\ &\hat{Y}_{(r-l+1),(k-l+1)}\} | X_{r-k}\Big) + R_{r,k}\end{aligned} \tag{9}$$
and n sufficiently small.
3) Using
$$\begin{aligned}&\{\hat{y}_{r,(N-1)}(u_{r,b-N+1}|x_{r-N+1}(s_{b-N+1})), \ldots \\ &\hat{y}_{(r-N+2),1}(u_{(r-N+2),(b-N+1)}|x_{r-N+1}(s_{b-N+1}))\} = \\ &\{k \in [1;N-1] : \\ &\hat{y}_{(r-k+1),(N-k)}(u_{(r-k+1),(b-N+1)}|x_{r-N+1}(s_{b-N+1})\}\end{aligned}$$
the destination now decodes $x_s(w_{b-N+1})$ iff
$$\begin{aligned}&\exists \tilde{w}_{b-N+1} : (\{k \in [1;N-1] : \\ &\hat{y}_{(r-k+1),(N-k)}(u_{(r-k+1),(b-N+1)}|x_{r-N+1}(s_{b-N+1})\}, \ldots \\ &y_d(b-N+1), x_{r-N+1}(s_{b-N+1}), x_s(\tilde{w}_{b-N+1})) \in A_\epsilon^{*(n)}.\end{aligned}$$
We can state that $\tilde{w}_{b-N+1} = w_{b-N+1}$ with arbitrarily low probability of error iff
$$\begin{aligned}R < &I\Big(X_s; Y_d, \{k \in [1;N-1] : \\ &\hat{Y}_{(r-k+1),(N-k)}\} | X_{r-N+1}\Big)\end{aligned} \tag{10}$$
and n sufficiently large.
4) Furthermore, all other relays decode the relay message $x_r(s_b)$ iff
$$R_r < I(X_r; Y_{r+k}), k \in [1;N-1] \tag{11}$$
and n sufficiently large. They further create the following tuple for $k \in [1;N-1]$
$$\left(y_{r+k}(b), \hat{y}_{(r+k),k}\left(u_{(r+k),b}|x_r(s_b)\right), x_r(s_b)\right) \in A_\epsilon^{*(n)}$$
which is possible iff
$$\hat{R}_{(r+k),k} > I(\hat{Y}_{r+k,k}, Y_{r+k}|X_r). \tag{12}$$

The previous points show that $B+N-1$ blocks are necessary to communicate B blocks, i.e., a rate loss $(N-1)/B \cdot R$ is implied which goes to 0 as $B \to \infty$. We can further reformulate (9) without loss of generality to
$$\begin{aligned}\hat{R}_{(r+k),k} < I\Big(&\hat{Y}_{(r+k),k}; Y_d, \{l \in [2;k] : \\ &\hat{Y}_{(r+k-l+1),(k-l+1)}\} | X_r\Big) + R_{(r+k),k}\end{aligned}$$

which implies using (12) that

$$I\left(\hat{Y}_{r+k,k}, Y_{r+k}|X_r\right) < I\left(\hat{Y}_{(r+k),k}; Y_d, \{l \in [2;k] : \hat{Y}_{(r+k-l+1),(k-l+1)}\}|X_r\right) + R_{(r+k),k},$$

and

$$R_{(r+k),k} > I\left(\hat{Y}_{(r+k),k}; Y_{r+k}|X_r, Y_d, \{l \in [2;k] : \hat{Y}_{(r+k-l+1),(k-l+1)}\}\right) \tag{13}$$

which is a result of the employed Wyner-Ziv coding. ■

Remark 1: The Markov lemma is necessary to show that the probability for

$$\left(x_s(w_b), x_r(s_b), \{r' \in \mathcal{R} \setminus \{r\} : \hat{y}_{r',(r'-r)}\left(u_b|x_r(s_b)\right)\}, \cdots \underline{y}_{\mathcal{R}\setminus\{r\}}(b), y_d(b)\right) \notin A_\epsilon^{*(n)}$$

is arbitrarily small if n is sufficiently large.

Appendix II
Proof of Theorem 2

Proof: We give here an outline of the proof for achievability of (4) by providing a random coding scheme achieving this rate. For this proof we require weak typicality as defined in [5, Ch. 14.2].

a) Random coding:

1) Define 2^{nR} conditionally i.i.d. n-length sequences $x_s(w|x_r(s))$ each with probability $p(x_s|x_r(s)) = \prod_{j=1}^{n} p(x_{sj}|x_r(s))$, $w \in \left[1; 2^{nR}\right]$, $r \in [1;N]$, $s \in \left[1; 2^{nR_r}\right]$. The dependence between source and relay message is used so that the source can assist the relay transmission, e.g., by coherent transmission.
2) Each relay creates a codebook consisting of 2^{nR_r} i.i.d. n-length sequences $x_r(s_r)$, $s_r \in \left[1; 2^{nR_r}\right]$, each with probability $p(x_r) = \prod_{j=1}^{n} p(x_{rj})$. Further let s_r be the vector of indices $\left(s_{r,1}, \cdots, s_{r,(N-1)}\right)$, $s_{r,k} \in \left[1; 2^{nR_{r,k}}\right]$ and $R_r = \sum_{k=1}^{N_1} R_{r,k}$.
3) Finally we introduce a random partitioning such that each $x_s(w|x_r(s))$ is randomly mapped into $2^{nR_{r,k}}$ cells $S_{r,k}(s_k)$, $s_k \in \left[1; 2^{nR_{r,k}}\right]$, $r \in [1;N]$, $k \in [1; N-1]$. We again use different rates to communicate each source symbol. Furthermore, we introduce the possibility that $R_{r,k} = 0$, i. e., the relay r transmitting in block b does not decode the source symbol $x_s(w_{b-k}|x_r(s_{b-k}))$.

b) Encoding: Consider block b in which relay r is transmitting. Assume $w_{b-k} \in S_{r,k}(s_{b,k})$ and relay r decoded all those w_{b-k} for which $R_{r,k} > 0$ correctly, it transmits $x_r(s_b)$ with $s_b = \left(s_{b,1}, \ldots, s_{b,(N-1)}\right)$ and the source transmits $x_s(w_b|x_r(s_b))$, $k \in [1; N-1]$ in block b. This is possible since the source node knows the mappings $S_{r,k}$ and therefore can determine by itself the message sent by the relay. Again, we assume that the previous $N-1$ steps were error free.

c) Decoding: At the end of block b the following decoding procedure is done (at the end of block $b > N-1$ the source index w_{b-N+1} is decoded):

1) The destination decodes $x_r(s_b)$ sent by relay r in block b. This is done by searching for a uniquely typical $x_r(s_b)$ with $y_d(b)$ which is possible iff $R_r < I(X_r; Y_d)$ and n sufficiently large.
2) In the next step the destination creates a set of those $\tilde{w}_{b-N+1}$ which can be the correct source index:
$$\mathcal{L}\left(y_d(b-N+1)\right) = \{\tilde{w}_{b-N+1} : (x_s(\tilde{w}_{b-N+1}|x_{r-N+1}(s_{b-N+1})), \cdots x_{r-N+1}(s_{b-N+1}), y_d(b-N+1)) \in A_\epsilon^{*(n)}\}.$$
Since the destination knows the bin indices it follows
$$\exists \tilde{w}_{b-N+1} : \tilde{w}_{b-N+1} = \mathcal{L}\left(y_d(b-N+1)\right) \cap \bigcap_{k=1}^{N-1} S_{(r-k+1),(N-k)}(s_{(b-k+1),(N-k)})$$
and $\tilde{w}_{b-N+1} = w_{b-N+1}$ with arbitrarily low probability of error iff
$$R < I(X_s; Y_d|X_{r-N+1}) + \sum_{k=1}^{N-1} R_{(r-k+1),(N-k)} \tag{14}$$
and n sufficiently large.
3) Let $r' = r+1$ be the next relay sending in block $b' = b+1$. At first r' needs to decode the vectors $x_{r'-k}(s_{(b'-k)})$, $k \in [1; N-1]$, which is possible with arbitrarily low probability of error iff $R_{r'-k} < I\left(X_{r'-k}; Y_{r'}\right)$ and n sufficiently large. Relay r' further creates the sets
$$\mathcal{L}_k\left(y_{r'}(b'-k)\right) = \{\tilde{w}_{b'-k} : (x_s(\tilde{w}_{b'-k}|x_{r'-k}(s_{b'-k})), \cdots y_{r'}(b'-k), x_{r'-k}(s_{b'-k})) \in A_\epsilon^{*(n)}\},$$
for $k \in [1; N-1]$. Using these sets and the previously decoded indices the relay now decodes iff
$$\exists \tilde{w}_{b'-k} : \tilde{w}_{b'-k} = \mathcal{L}_k\left(y_{r'}(b'-k)\right) \cap \bigcap_{l=1}^{k-1} S_{(r'-l),(k-l)}(s_{(b'-l),(k-l)})$$
which succeeds (for all k such that $R_{r',k} > 0$) such that $\tilde{w}_{b'-k} = w_{b'-k}$ iff
$$R < I(X_s; Y_{r'}|X_{r'-k}) + \sum_{l=1}^{k-1} R_{r'-l,k-l} \tag{15}$$
and n sufficiently large.

■