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A Generalized Mixed Strategy for Multiterminal Relay Networks

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***Abstract*—In their fundamental paper, Cover and El Gamal presented three basic coding strategies – *decode-and-forward*, *compress-and-forward* and a *mixed strategy* based on *partial decode-and-forward* and compress-and-forward – which are still the basis for many recent relaying protocols. So far, only parts of their work are applied to networks of relay nodes, e. g., the decode-and-forward as well as compress-and-forward approach. This work generalizes a mixed approach of partial decode-and-forward and compress-and-forward to networks of relay nodes. We further highlight how the "successive refinement problem" and the "broadcast channel problem with degraded message sets" are applied in our approach. Finally, we formulate achievable rates for the discrete memoryless relay channel consisting of two relay nodes.**

I. Introduction and motivation

We can observe a growing importance of infrastructure based wireless communications systems as well as ad hoc networks in present-day telecommunications. The popularity of mobile terminals poses the question how to exploit a network of wireless terminals to increase for instance capacity and coverage or to reduce usage of backhaul infrastructure. One answer to this question is to use *relay nodes* supporting the end-to-end communication between two nodes.

This idea of relaying goes back to van der Meulen [1], [2]. Cover and El Gamal refined this idea in [3] and presented three basic coding strategies for the three-terminal case: *decode-and-forward* (DF), *compress-and-forward* (CF) as well as a *mixed strategy* combining *partial decode-and-forward* and compress-and-forward. In recent publications the analysis of the three-terminal case was extended to the multi-terminal case: Kramer *et al.* presented in [4] different coding strategies for networks of relay nodes, e. g., a generalized DF and CF for relay networks as well as a mixed strategy where each relay node uses either DF or CF. Gupta and Kumar generalized in [5] the DF approach presented in [3] to a multi-level relaying scenario which serves as the basis for our proposal. However, to the authors' knowledge, no strategy was published so far which generalizes the mixed strategy based on partial DF to a relay network such that each node operates in a mixed mode.

We apply in this work the approach of [5] to the mixed strategy presented in [3, Theorem 7] for the three-terminal case. The mixed strategy based on partial DF divides the source message into two parts: U_1 and U_2 where the former one can be decoded without knowledge of U_2. As illustrated by the information flow in Fig. 1, relay node r only decodes the first source message U_1 for which it selects a message index using the random binning procedure introduced in [6]. This index determines the support message V transmitted by the relay node (which is also known to the source node). Using $\hat{\mathrm{Y}}_r$ the relay node quantizes the remaining uncertainty about U_2 in its channel output Y_r. This quantization is used to select some index which determines the second relay message W.

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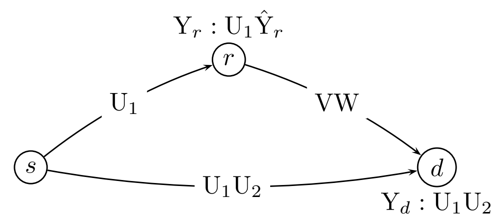

Fig. 1. The information flow of the mixed approach using partial decode-and-forward. Arc labels show which information is exchanged between nodes and node labels show the channel outputs and decoded symbols.

The destination node d decodes the support message V which provides redundant information such that the source message U_1 can be decoded. Using the message W and the correlation between the relay and destination channel output the destination decodes the quantization $\hat{\mathrm{Y}}_r$ (a strategy similar to Wyner-Ziv coding [7], [8]). With this quantization the destination can decode the second source message U_2.

Our approach now generalizes this protocol by introducing $N+1$ degraded source messages. Each message U_k is only decoded by a subset of all relay nodes whereas each node l of this set determines a supporting message V_l^k. Furthermore, these nodes transmit successively refined quantizations of their channel output to those relays which decode the messages $\mathrm{U}_{k'}$, $k' > k$. After a detailed definition of our nomenclature and used network model in Section II, we describe this general approach in more detail in Section III. Finally, we conclude the paper with an outlook in Section IV.

II. Relay network model and nomenclature

In the following we will use non-italic uppercase letters X to denote random variables, non-italic lowercase letters x to denote events of a random variable and italic letters (N

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or n) are used to denote scalars. Ordered sets are denoted by $\mathcal{X}$, the cardinality of an ordered set is denoted by $\|\mathcal{X}\|$ and $[b; b+k]$ is used to denote the ordered set of numbers $(b, b+1, \cdots, b+k)$. Let X_k be a random variable parameterized using k then $\mathbf{X}_{\mathcal{C}}$ denotes the vector of all X_k with $k \in \mathcal{C}$ (this applies similarly to sets of events). Furthermore, we will use in the following $p(\mathrm{x}|\mathrm{y})$ to abbreviate the conditional pdf $p_{\mathrm{X}|\mathrm{Y}}(\mathrm{x}|\mathrm{y})$ if this does not create any confusion.

This paper considers a network of $N+2$ nodes: the set of N relays $t \in \mathcal{R} := [1; N]$, the destination node $d = N+1$ and the source node $s = N+2$. The discrete memoryless relay channel is defined by the conditional pdf $p\left(\mathbf{y}_{\mathcal{R}}, \mathrm{y}_d | \mathrm{x}_s, \mathbf{x}_{\mathcal{R}}\right)$ over all possible channel inputs $(\mathrm{x}_1, \cdots, \mathrm{x}_N, \mathrm{x}_s) \in \mathcal{X}_1 \times \cdots \mathcal{X}_N \times \mathcal{X}_s$ and channel outputs $(\mathrm{y}_1, \cdots, \mathrm{y}_N, \mathrm{y}_d) \in \mathcal{Y}_1 \times \cdots \mathcal{Y}_N \times \mathcal{Y}_d$ with $\mathcal{X}_i$ and $\mathcal{Y}_j$ denoting the input and output alphabets.

Remark 1: In comparison to the multi-level approach in [5] we concentrate in this paper on the case that each level/group consists of one relay node. Section III-D addresses upcoming issues if we group the relay nodes to disjoint sets.

Let $\pi(\mathcal{X})$ be the set of all permutations of a set $\mathcal{X}$. The source chooses an ordering $o_s \in \pi([1; N+1])$ where $o_s(l)$ denotes the l-th element of o_s and $o_s(N+1) = N+1$. For the sake of readability we abbreviate in the following $\mathrm{Y}_{o_s(l)}$ by Y_l and the relay node $o_s(l)$ by l or as the *l-th level*. Besides, each relay l introduces an ordering $o_l \in \pi([l+1; N+1])$ where $o_l(i)$ indicates node $o_s(o_l(i))$ and the channel output of this node is denoted by $\mathrm{Y}_{l,i}$. We further use in the following the function ϕ_l to denote the inverse of o_l, i. e., $o_l(\phi_l(i)) = i$.

III. A GENERALIZED MIXED PARTIAL DF APPROACH

In our proposal the following messages are considered:

- the source messages U_k, $k \in [1; N+1]$, with rates R_s^k,
- the messages V_l^k sent by level l at rate R_l^k to assist U_k,
- the quantizations $\hat{\mathrm{Y}}_l^{k'}$, $k' \in [1; M_l]$ with $M_l = N - l + 1$, and the corresponding broadcast messages $\mathrm{W}_l^{k'}$.

The partial message U_k is decoded by each level $l \geq k$. To support the source message the relays, i. e., $l \leq N$, assign to each partial message an index V_l^k obtained using random binning [6] and transmit it to each level $l' > l$. Levels $l' > l$ use V_l^k to decode the respective partial source message.

Furthermore, each relay level l estimates the remaining uncertainty in its receive vector using the quantizations $\hat{\mathrm{Y}}_l^{k'}$, $k' \in [1; M_l]$ at distortion $D_l^{k'} = d(\hat{\mathrm{Y}}_l^{k'}, \mathrm{Y}_l)$ where $d(\cdot, \cdot)$ denotes a suitable distortion measure and $D_l^{k'} > D_l^{k'+1}$ holds. These quantizations are assigned to the messages $\mathrm{W}_l^{k'}$. Level $o_l(i)$, $i \in [1; M_l]$, decodes all messages $\mathbf{W}_l^{[1;i]}$ and quantizations $\hat{\mathbf{Y}}_l^{[1;i]}$, respectively. We will further explain in the sequel the mentioned quantization task, which is known as the successive refinement problem with multiple descriptors and (in general) unstructured side information. Besides, the messages W_l^k need to be transmitted to the respective levels which is similar to the broadcast channel problem.

A. The successive refinement problem

Koshelev [9] and later Equitz and Cover [10] independently introduced the problem of successive refinement as a special case of the more general multiple description problem [11]. Fig. 3 illustrates the successive refinement problem as it emerges in our proposal. The channel output of level l, i. e., Y_l, has to be encoded and transmitted to nodes $o_l(i)$, $i \in [1; M_l]$. At first the channel output is estimated by a quantization $\hat{\mathrm{Y}}_l^1$ using $\Delta_l^1 = R_{\mathrm{Y}_l}(D_l^1)$ bits per symbol, where $R_{\mathrm{Y}_l}(D)$ is the rate-distortion function for some given distortion D. This estimation needs to be decoded by all nodes $o_l(i)$. Since these nodes can exploit (in general unstructured) side information $\mathrm{Y}_{l,i}$, the necessary rate $\hat{R}_l^1 \leq \Delta_l^1$ to describe Y_l at distortion D_l^1 is the well known Wyner-Ziv source coding problem [7], [8], i. e., $\hat{R}_l^1 = \max_{i \in [1; M_l]} R^{\mathsf{WZ}}_{\mathrm{Y}_l|\mathrm{Y}_{l,i}}(D_l^1)$, where $R^{\mathsf{WZ}}_{\cdot|\cdot}(\cdot)$ is the Wyner-Ziv rate-distortion function as defined in [8]. In the next refinement step all levels $o_l(i)$, $i \geq 2$, additionally decode the more accurate description $\hat{\mathrm{Y}}_l^2$ with $D_l^2 < D_l^1$. To describe the refined quantization $\hat{\mathrm{Y}}_l^2$ additional information at rate $\hat{R}_l^2$ must be provided. Again from rate-distortion theory we know that $\hat{R}_l^2 + \hat{R}_l^1 \geq \max_{i \in [2; M_l]} R^{\mathsf{WZ}}_{\mathrm{Y}_l|\mathrm{Y}_{l,i}}(D_l^2)$.

In [9], [10] the Markovity condition to achieve rate-distortion optimal successive refinements is derived. In our

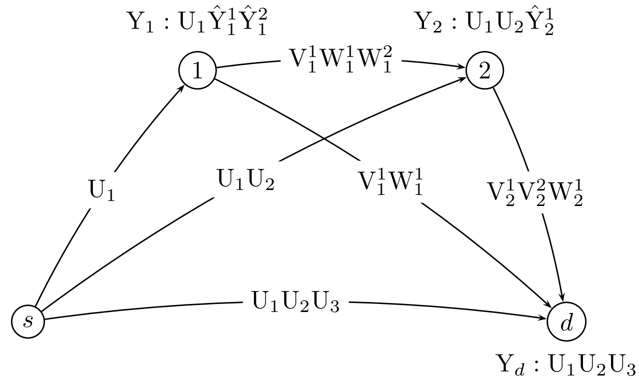

Fig. 2. The information flow in our proposal for $N = 2$. The source transmits in order $o_s = \{1, 2, d\}$ the partial messages $\mathrm{U}_{[1;3]}$ and the first relay chooses $o_l = \{d, 2\}$ for the quantization of the receive vector.

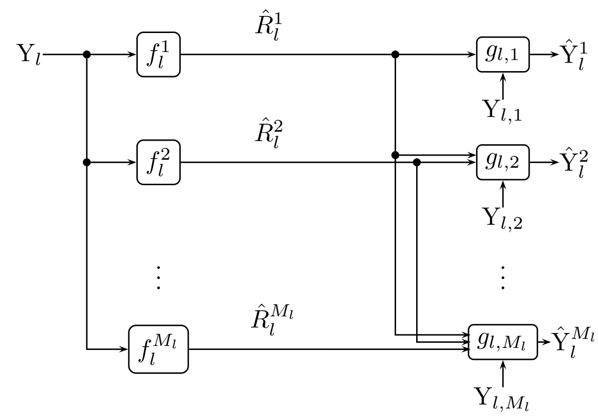

Fig. 3. The successive refinement problem as it emerges in our protocol. Y_l needs to be quantized by encoders $f_{l,1}, \cdots, f_{l,M_l}$ at different distortions and to be decoded by decoders $g_{l,1}, \cdots, g_{l,M_l}$, $M_l = N - l + 1$, which can exploit unstructured side information $\mathbf{Y}_{l,[1;M_l]}$.

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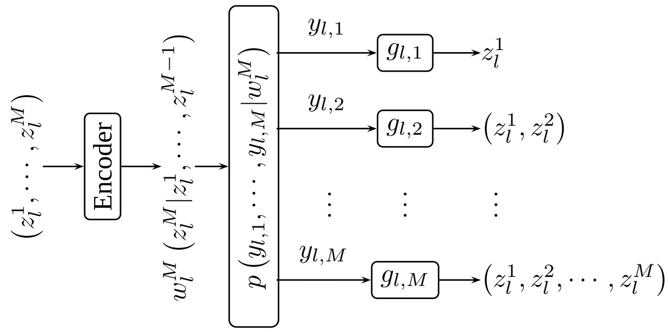

Fig. 4. The broadcast channel problem considered in our work. The indices $\mathbf{z}_l^{[1;k]}$ are determined by the Wyner-Ziv coding of the quantizations $\hat{\mathbf{Y}}_l^{[1;k]}$ and need to be decoded by receiver $g_{l,k}, \cdots, g_{l,M_l}$, $M_l = N - l + 1$.

setting this implies the Markov chain $\mathrm{Y}_l \leftrightarrow \hat{\mathrm{Y}}_l^{M_l} \leftrightarrow \hat{\mathrm{Y}}_l^{M_l-1} \leftrightarrow \cdots \hat{\mathrm{Y}}_l^1$. With this condition we can ensure that $\hat{\mathrm{Y}}_l^{k'}$ is at least for one node $o_l(i)$, $i \geq k'$, rate-distortion optimal, i.e., $\sum_{i=1}^{k'} \hat{R}_l^i = \max_{i \in [k';M_l]} R^{\mathrm{WZ}}_{\mathrm{Y}_l|\mathrm{Y}_{l,i}}(D_l^{k'})$. For the case of no side information, Equitz and Cover show in [10] that this Markovity condition is necessary and sufficient for rate-distortion optimality in each refinement step. Our setup is a special case of the general setting presented by Heegard and Berger in [12, Theorem 2] for which an upper bound on the rate region is known. So far it is unresolved whether this bound is tight.

B. The broadcast channel problem

In this subsection we treat the difficulties of transmitting the quantizations of level l, i.e., $\hat{\mathbf{Y}}_l^{[1;M_l]}$, to the next M_l levels. Fig. 4 illustrates the problem: the message indices $(z_l^1, \ldots, z_l^{M_l})$, $z_l^{k'} \in [1; 2^{n\hat{R}_l^{k'}}]$, are determined by the quantizations $\hat{\mathrm{Y}}_l^1, \ldots, \hat{\mathrm{Y}}_l^{M_l}$ using a random binning procedure. As previously mentioned, nodes $o_l(i)$, $i \in [k'; M_l]$, need the indices $(z_l^1, \ldots, z_l^{k'})$ to successfully decode the quantizations $\hat{\mathrm{Y}}_l^1, \ldots, \hat{\mathrm{Y}}_l^{k'}$. Obviously, this is a broadcasting problem as introduced by Blackwell and later described in detail by Cover [13]. Our problem is characterized by a degraded message set $\mathbf{W}_l^{[1;M_l]}$, which was analyzed by Körner and Marton [14]. Later, Csiszar and Körner further investigated this problem as the “asymmetric broadcast channel” [15].

To communicate the message indices $\mathbf{z}_l^{[1;M_l]}$, relay level l transmits the messages $\mathrm{W}_l^{k'}$, $k' \in [1; M_l]$, which in our proposal build the Markov chain $\mathrm{W}_l^1 \leftrightarrow \mathrm{W}_l^2 \leftrightarrow \ldots \mathrm{W}_l^{M_l} \leftrightarrow (\mathrm{Y}_{l,1}, \ldots, \mathrm{Y}_{l,M_l})$ (this is not a necessary condition but it simplifies the expressions in the sequel). Using the results of [14] we can intuitively state that

$$\hat{R}_l^k \leq \min_{i \in [k;M_l]} I\left(\mathrm{W}_l^k; \mathrm{Y}_{l,i} | \mathrm{W}_l^{k-1}\right) \tag{1}$$

is an achievable rate region for our problem. As explained in [16, Corollary 5], (1) is included in the capacity region for the case of $M_l = 2$: $\hat{R}_l^1 \leq I(\mathrm{W}_l^1; \mathrm{Y}_{l,1})$, $\hat{R}_l^2 \leq I(\mathrm{W}_l^2; \mathrm{Y}_{l,2}|\mathrm{W}_l^1)$ and $\hat{R}_l^1 + \hat{R}_l^2 \leq I(\mathrm{W}_l^2; \mathrm{Y}_{l,2})$ [14]. In the special case that $\mathrm{Y}_{l,2}$ is not “less noisy” than $\mathrm{Y}_{l,1}$ [14], i.e., $I(\mathrm{W}_l^1; \mathrm{Y}_{l,1}) > I(\mathrm{W}_l^1; \mathrm{Y}_{l,2})$, we need to introduce a time-sharing and auxiliary random variable to achieve capacity [15]. Since the generalization of this method to prove the capacity region of our setting is beyond the scope of this paper we use (1) in the sequel.

C. Achievable rates for $N = 2$

In this subsection we apply the previously presented results to describe the encoding and decoding procedure of our proposal in more detail for the case of $N = 2$ relay nodes. The application of this proposal to $N > 2$ relay terminals is straightforward but unnecessarily complicates the derivations.

Basically, our protocol represents a multi-hop implementation of the mixed strategy based on partial decode-and-forward presented in [3, Theorem 7]. Consider the partial source messages $\mathbf{U}_{[1;3]}$ transmitted in block b. Level 1 decodes U_1 and uses it to determine the message V_1^1 by assigning to each possible source message a bin index. This bin index is used to select the relay message V_1^1 (a well known method introduced by Slepian-Wolf [6]). This random binning is slightly different for the next level 2: instead of assigning the index randomly to the source messages it assigns the indices to the bins of the first relay level. Since level 2 supports in block $b+1$ the same source message as level 1 in block b, we can ensure that level 1 knows in each block the message V_2^1 [5]. Using the decoded messages V_1^1 and U_1 level 1 quantizes the remaining uncertainty in the receive vector Y_1 using the quantizations $\hat{\mathbf{Y}}_1^{[1;2]}$ which determine the broadcast messages $\mathbf{W}_1^{[1;2]}$.

In the next block $(b+1)$ relay 2 decodes the supporting message V_1^1 and uses the additional information to decode U_1 sent in block b. With this knowledge the second relay can decode the quantization indices $\mathbf{W}_1^{[1;\phi(1,2)]}$ and the quantization dedicated to level 2, i.e., $\hat{\mathrm{Y}}_1^{\phi_1(2)}$. Using this quantization and the own channel output Y_2 in block b the relay is able to decode the second partial source message U_2. Knowing this partial message and the indices transmitted by the previous relay level, it further quantizes the remaining uncertainty in Y_2 by $\hat{\mathrm{Y}}_2^1$ and determines W_2^1. Furthermore, this level assigns the supporting messages $\mathbf{V}_2^{[1;2]}$ to both source messages where V_2^1 is determined using the set inclusion method described above.

Finally, the destination decodes in block $b+2$ at first the message V_2^1 and uses it to decode message V_1^1 transmitted in block $b+1$. That followed, the destination is able to decode the first partial source message U_1 sent in block b. In the next step, it decodes the relay message V_2^2, the quantization $\hat{\mathrm{Y}}_1^{\phi_1(3)}$ and using both also the source message U_2. Finally, it uses the quantizations $\hat{\mathrm{Y}}_1^{\phi_1(3)}$, $\hat{\mathrm{Y}}_2^1$ and the channel output Y_d in block b to decode the source message U_3 sent in block b. The information flow of this proposal is shown in Fig. 2. A more detailed description of a random coding scheme is given in the Appendix which outlines the proof of the following theorem.

Theorem 1: The final rate $R = \sum_{l=1}^3 R_s^l$ is given by

$$R = \max_{o_s \in \pi([1;3])} \max_{o_1 \in \pi([2;3])} \sup_p \sum_{k=1}^3 R_s^k \tag{2}$$

where the maximum is taken over all possible o_s and o_1. The rates of the partial messages are given by

$$\begin{aligned} R_s^1 < \min\big(& I(\mathrm{U}_1;\mathrm{Y}_1|V_2^1,W_1^2), \\ & R_1^1 + \min_{l\in[2;3]} I(\mathrm{U}_1;\mathrm{Y}_l|\mathrm{W}_1^{\phi_1(l)},\mathrm{W}_2^1)\big), \end{aligned} \quad (3)$$

$$\begin{aligned} R_s^2 < \min\big(& I(\mathrm{U}_2;\mathrm{Y}_2\hat{\mathrm{Y}}_1^{\phi_1(2)}|\mathbf{W}_{[1;2]}^1\mathrm{W}_1^{\phi_1(2)}\mathrm{V}_2^1\mathrm{U}_1), \\ & I(\mathrm{U}_2;\mathrm{Y}_3\hat{\mathrm{Y}}_1^{\phi_1(3)}|\mathbf{W}_{[1;2]}^1\mathrm{W}_1^{\phi_1(3)}\mathrm{V}_2^1\mathrm{U}_1) + I\left(\mathrm{V}_2^2;\mathrm{Y}_3|\mathrm{V}_2^1\right)\big) \end{aligned} \quad (4)$$

and

$$R_s^3 < I\left((\mathrm{U}_3;\mathrm{Y}_3\hat{\mathrm{Y}}_1^{\phi_1(3)}\hat{\mathrm{Y}}_2^1|\mathrm{U}_2\mathrm{U}_1\mathbf{W}_{[1;2]}^1\mathrm{W}_1^{\phi_1(3)}\mathrm{V}_2^1\right), \quad (5)$$

subject to the side condition on the first relay transmission

$$R_1^1 < \min\left(I\left(\mathrm{V}_1^1;\mathrm{Y}_2|\mathrm{W}_2^1\right), I\left(\mathrm{V}_1^1;\mathrm{Y}_3|\mathrm{W}_2^1\right) + I\left(\mathrm{V}_2^1;\mathrm{Y}_3\right)\right), \quad (6)$$

the successive refinement conditions for level 1

$$\hat{R}_1^1 > \max_{l\in[1;2]} I(\hat{\mathrm{Y}}_1^1;\mathrm{Y}_1|\mathrm{V}_2^1\mathbf{W}_1^{[1;l]}\mathrm{U}_1\mathrm{Y}_{1,l}\mathrm{W}_2^1) \quad (7)$$

$$\hat{R}_1^1 + \hat{R}_1^2 > I(\hat{\mathrm{Y}}_1^2;\mathrm{Y}_1|\mathrm{V}_2^1\mathbf{W}_1^{[1;2]}\mathrm{W}_2^1\mathrm{U}_1\mathrm{Y}_{1,2}), \quad (8)$$

the Wyner-Ziv condition on the quantizations of level 2

$$\begin{aligned} & I\left(\hat{\mathrm{Y}}_2^1;\mathrm{Y}_2|\mathbf{W}_{[1;2]}^1\mathrm{U}_2\right) < I(\mathrm{W}_2^1;\mathrm{Y}_3|\mathrm{V}_2^2) \\ & + \begin{cases} I(\mathrm{Y}_3\hat{\mathrm{Y}}_1^2;\hat{\mathrm{Y}}_2^1|\mathrm{V}_2^1\mathbf{W}_{[1;2]}^1\mathbf{U}_{[1;2]}), & \text{if } \phi_1(3)=1, \\ I(\mathrm{Y}_3\hat{\mathrm{Y}}_1^2\mathrm{W}_1^2;\hat{\mathrm{Y}}_2^1|\mathrm{V}_2^1\mathbf{W}_{[1;2]}^1\mathbf{U}_{[1;2]}), & \text{otherwise,} \end{cases} \end{aligned} \quad (9)$$

and the broadcast conditions

$$\hat{R}_1^1 < \min_{l\in[1;2]} I(\mathrm{W}_1^1;\mathrm{Y}_{1,l}|\mathrm{V}_1^1,\mathrm{W}_2^1) \quad (10)$$

$$\hat{R}_1^2 < I(\mathrm{W}_1^2;\mathrm{Y}_{1,2}|\mathbf{W}_{[1;2]}^1). \quad (11)$$

The supremum in (2) is taken over all joint pdf

$$\begin{aligned} & p(\mathrm{u}_{[1;3]},\mathrm{v}_1^1,\mathrm{w}_1^{[1;2]},\hat{\mathrm{y}}_1^{[1;2]},\mathrm{v}_2^{[1;2]},\mathrm{w}_2^1,\hat{\mathrm{y}}_2^1,\mathrm{y}_{[1;3]}) = p(\mathrm{u}_{[1;3]}) \\ & \cdot p(\mathrm{v}_1^1,\mathrm{w}_1^{[1;2]})p(\hat{\mathrm{y}}_1^{[1;2]}|\mathrm{y}_1,\mathrm{u}_1,\mathrm{w}_1^1,\mathrm{v}_2^1)p(\mathrm{v}_2[1;2],\mathrm{w}_2^1) \\ & \cdot p(\hat{\mathrm{y}}_2^1|\mathrm{y}_2,\mathrm{u}_2,\mathrm{w}_2^1,\mathrm{w}_1^1)p(\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3|\mathrm{u}_3,\mathrm{w}_1^2,\mathrm{w}_2^1) \end{aligned} \quad (12)$$

with

$$p(\mathrm{u}_{[1;3]}) = p(\mathrm{u}_1)\prod_{k=2}^3 p(\mathrm{u}_k|\mathrm{u}_{k-1}) \quad (13)$$

$$p(\mathrm{v}_1^1,\mathrm{w}_1^{[1;2]}) = p(\mathrm{w}_1^2|\mathrm{w}_1^1)p(\mathrm{w}_1^1|\mathrm{v}_1^1)p(\mathrm{v}_1^1) \quad (14)$$

$$p(\mathrm{v}_2[1;2],\mathrm{w}_2^1) = p(\mathrm{w}_2^1|\mathrm{v}_2^2)p(\mathrm{v}_2^2|\mathrm{v}_2^1)p(\mathrm{v}_2^1) \quad (15)$$

$$\begin{aligned} p(\hat{\mathrm{y}}_1^{[1;2]}|\mathrm{y}_1,\mathrm{u}_1,\mathrm{w}_1^1,\mathrm{v}_2^1) =& p(\hat{\mathrm{y}}_1^1|\hat{\mathrm{y}}_1^2,\mathrm{u}_1,\mathrm{w}_1^1,\mathrm{v}_2^1) \\ & \cdot p(\hat{\mathrm{y}}_1^2|\mathrm{y}_1,\mathrm{u}_1,\mathrm{w}_1^1,\mathrm{v}_2^1). \end{aligned} \quad (16)$$

Proof: Outlined in the Appendix. ■

D. Further problems

Consider the case that the relay nodes and destination are clustered into $L+1$ disjoint sets $\mathcal{R}_l$, $l \in [1;L+1]$, with $\mathcal{R}_{L+1} = \{d\}$. For the sake of simplicity we considered in the previous discussion that $\|\mathcal{R}_l\| = 1$ and $N = L$. Nevertheless, the extension to $\|\mathcal{R}_k\| > 1$ and $L < N$ is straightforward but involves additional considerations. Assume for instance that each relay node per level is concurrently transmitting in the way defined above. In this case we have to consider a multiple access channel between each level set [17, Ch. 14.3], but the actual protocol design does not change. Another problem to be considered is the half-duplex constraint which implies that each relay node is only able to either transmit or receive on the same time-frequency resource [18], [19]. Among others [20], [21] present a possible extension if $\|\mathcal{R}_k\| > 1$ where the relay nodes alternately transmit. Note, each transmitting group within one $\mathcal{R}_k$ can consists of more than one node which combines the problems of the multiple access channel and the half-duplex constraint. Obviously, these problems complicate the analysis but do not affect the basic proposal.

IV. Conclusion

In this work we presented a generalization of the mixed partial decode-and-forward protocol to a multi-level scenario. This generalization faces two problems: the successive refinement problem and the broadcast channel problem. Known solutions to these problems are applied to show achievable rates for the case of two levels. Furthermore, a brief outline considered additional problems if $\|\mathcal{R}_k\| > 1$ and if the half-duplex constraint is applied. Further work will include the application to wireless models, e. g., the Gaussian relay channel and fading channels, as well as the application to certain coding schemes which are more practical to use.

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APPENDIX

This appendix gives an outline for the proof of the achievable rates given in Theorem 1. We describe in the following a random coding scheme, the encoding as well as the decoding procedure for $N = 2$. The proof relies on the definition of strongly typical sets $\mathcal{A}_\epsilon^{*(n)}$ which is for instance given in [17, Ch. 13.6]. A more detailed proof of the probability of error in B blocks follows standard arguments which are extensively used in previous work, e. g., [3], [5].

a) ***Random Coding****:* At first we describe the random codes where $\mathrm{x}(i) \overset{n}{\sim} p(\mathrm{x})$ denotes that the n-length sequences $\mathrm{x}(i)$ are drawn i.i.d. according to $p(\mathrm{x}) = \prod_{j=1}^n p(\mathrm{x}_j)$ where x_j denotes the j-th element of the n-length sequence x. The source generates the following n-length sequences:

$$\begin{aligned}\mathrm{u}_1(q^1) &\overset{n}{\sim} p(\mathrm{u}_1), q^1 \in [1; 2^{nR_1^1}],\\ \mathrm{u}_2(q^2|q^1) &\overset{n}{\sim} p(\mathrm{u}_2|\mathrm{u}_1(q^1)), q^2 \in [1; 2^{nR_1^2}],\\ \mathrm{u}_3(q^3|\mathbf{q}^{[1;2]}) &\overset{n}{\sim} p(\mathrm{u}_3|\mathrm{u}_2(q^2|q^1)), q^3 \in [1; 2^{nR_1^3}].\end{aligned}$$

Note that the Markov chain $\mathrm{U}_1 \leftrightarrow \mathrm{U}_2 \leftrightarrow \mathrm{U}_3$ is again not a necessary condition but a simplification of the following derivations. Relay level 1 generates the following n-length sequences

$$\begin{aligned}\mathrm{v}_1^1(s_1^1) &\overset{n}{\sim} p(\mathrm{v}_1^1), s_1^1 \in [1; 2^{nR_1^1}],\\ \mathrm{w}_1^1(z_1^1|s_1^1) &\overset{n}{\sim} p(\mathrm{w}_1^1|\mathrm{v}_1^1(s_1^1)), \mathrm{w}_1^1 \in [1; 2^{n\hat{R}_1^1}],\\ \mathrm{w}_1^2(z_1^2|s_1^1, \mathrm{w}_1^1) &\overset{n}{\sim} p(\mathrm{w}_1^2|\mathrm{w}_1^1(z_1^1|s_1^1)), \mathrm{w}_1^2 \in [1; 2^{n\hat{R}_1^2}].\end{aligned}$$

Furthermore, level 2 generates

$$\begin{aligned}\mathrm{v}_2^1(s_2^1) &\overset{n}{\sim} p(\mathrm{v}_2^1), s_2^1 \in [1; 2^{nR_2^1}],\\ \mathrm{v}_2^2(s_2^2|s_2^1) &\overset{n}{\sim} p(\mathrm{v}_2^2|\mathrm{v}_2^1(s_2^1)), s_2^2 \in [1; 2^{nR_2^2}],\\ \mathrm{w}_2^1(z_2^1|\mathbf{s}_2^{[1;2]}) &\overset{n}{\sim} p(\mathrm{w}_2^1|\mathrm{v}_2^2(s_2^2|s_2^1)), \mathrm{w}_2^1 \in [1; 2^{n\hat{R}_2^1}].\end{aligned}$$

To implement the previously described successive refinement both relay levels need to generate

$$\begin{aligned}\hat{\mathrm{y}}_1^1(r_1^1|q^1, \mathbf{s}_{[1;2]}^1, z_1^1) &\overset{n}{\sim} p(\hat{\mathrm{y}}_1^1|\mathrm{u}_1(q^1), \mathrm{w}_1^1(z_1^1|s_1^1), \mathrm{v}_2^1(s_2^1),)\\ \hat{\mathrm{y}}_2^1(r_2^1|\mathbf{q}^{[1;2]}, \mathbf{s}_{[1;2]}^1, s_2^2, \mathbf{z}_{[1;2]}^1) &\overset{n}{\sim} p(\hat{\mathrm{y}}_2^1|\mathrm{u}_2(q^2|q^1), \mathrm{w}_1^1(z_1^1|s_1^1),\\ &\qquad \mathrm{w}_2^1(z_2^1|\mathbf{s}_2^{[1;2]}),)\\ \hat{\mathrm{y}}_1^2(r_1^2|r_1^1, q^1, \mathbf{s}_{[1;2]}^1, z_1^1) &\overset{n}{\sim} p(\hat{\mathrm{y}}_2^1|\hat{\mathrm{y}}_1^1(r_1^1|q^1, \mathbf{s}_{[1;2]}^1, z_1^1),\\ &\qquad \mathrm{u}_1(q^1), \mathrm{w}_1^1(z_1^1|s_1^1), \mathrm{v}_2^1(s_2^1)),\end{aligned}$$

with $r_k^1 \in [1; 2^{n\Delta_k^1}]$, $k \in [1; 2]$, $r_1^2 \in [1; 2^{n(\Delta_1^2 - \Delta_1^1)}]$. The used pdf's are obtained by appropriate manipulation of the joint pdf given in (12).

Remark 2: Note that neither the source transmission depends on the relay transmission nor do the transmission of relay level l depend on messages sent by levels $l' > l$ although this is possible. We forbear from introducing this additional coding feature for the sake of simplicity.

Finally we need to define the following random partitions

- Relay level 1 defines the sets $\mathcal{S}_1^1(s_1^1)$, $s_1^1 \in [1; 2^{nR_1^1}]$, by randomly assigning each index q^1 to one of these sets according to a uniform distribution over the indices s_1^1. Level 2 creates the sets $\mathcal{S}_2^1(s_2^1)$, $s_2^1 \in [1; 2^{nR_2^1}]$, by randomly and uniformly assigning each of the indices s_1^1 to one of the sets. This set inclusion algorithm was introduced by Gupta and Kumar in [5].
- Relay level 2 further defines the sets $\mathcal{S}_2^2(s_2^2)$, $s_2^2 \in [1; 2^{nR_2^2}]$, by randomly and uniformly assigning each index q^2 to one of these sets.
- To implement the described successive refinement problem with side information relay level 1 randomly and uniformly assigns each index r_1^k, $k \in [1; 2]$, to one of the sets $\mathcal{Z}_1^k(z_1^k)$, $z_1^k \in [1; 2^{n\hat{R}_1^k}]$. In the same way level 2 randomly and uniformly assigns each index r_2^1 to one of the sets $\mathcal{Z}_2^1(z_2^1)$, $z_2^1 \in [1; 2^{n\hat{R}_2^1}]$.

b) ***Encoding****:* Let the source transmit in block b the messages $\mathrm{u}_1(q_b^1)$, $\mathrm{u}_2(q_b^2|q_b^1)$ and $\mathrm{u}_3(q_b^3|\mathbf{q}_b^{[1;2]})$. Further assume that the decoding in the previous two blocks was error free at both relay nodes and the destination. Now let

- $q_{b-1}^1 \in \mathcal{S}_1^1(s_{1,b}^1)$, $s_{1,b-1}^1 \in \mathcal{S}_2^1(s_{2,b}^1)$ and $q_{b-2}^2 \in \mathcal{S}_2^2(s_{2,b}^2)$,
- $r_{1,b-1}^1 \in \mathcal{Z}_1^1(z_{1,b}^1)$, $r_{1,b-1}^2 \in \mathcal{Z}_1^2(z_{1,b}^2)$ and $r_{2,b-2}^1 \in \mathcal{Z}_2^1(z_{2,b}^1)$.

We rigorously define in the sequel how these indices are obtained. Using these indices, level 2 transmits the message $\mathrm{w}_1^2(z_{1,b}^2|z_{1,b}^1, s_{1,b}^1)$ and the second level $\mathrm{w}_2^1(z_{2,b}^1|\mathbf{s}_{2,b}^{[1;2]})$.

c) ***Decoding****:* The decoding at **the first relay level** is done as follows

- Using the channel output in block b, i. e., $\mathrm{y}_1(b)$, the relay decodes $\mathrm{u}_1(q_b^1)$ given $\mathrm{w}_1^2(z_{1,b}^2|z_{1,b}^1, s_{1,b}^1)$ and $\mathrm{v}_2^1(s_{2,b}^1)$ (obtained by knowing $s_{1,b-1}^1$). Obviously this can be done almost error free iff n is sufficiently large and
$$R_s^1 < I\left(\mathrm{U}_1; \mathrm{Y}_1|\mathrm{V}_2^1, \mathrm{W}_1^2\right). \tag{17}$$
- Knowing $\mathbf{s}_{[1;2],b}^1$, $z_{1,b}^1$, q_b^1 it selects $r_{1,b}^{[1;2]}$ such that
$$\Big(\mathrm{y}_1(b), \hat{\mathrm{y}}_1^1\left(r_{1,b}^1|s_{2,b}^1, s_{1,b}^1, z_{1,b}^1, q_b^1\right), \mathrm{v}_2^1\left(s_{2,b}^1\right), \mathrm{w}_1^1(z_{1,b}^1|s_{1,b}^1), \mathrm{u}_1\left(q_b^1\right)\Big) \in \mathcal{A}_\epsilon^{*(n)}$$
and
$$\Big(\mathrm{y}_1(b), \hat{\mathrm{y}}_1^2\left(r_{1,b}^2|r_{1,b}^1, s_{2,b}^1, s_{1,b}^1, z_{1,b}^1, q_b^1\right), \mathrm{v}_2^1\left(s_{2,b}^1\right), \hat{\mathrm{y}}_1^1\left(r_{1,b}^1|s_{2,b}^1, z_{1,b}^2, q_b^1\right), \mathrm{w}_1^1(z_{1,b}^1|s_{1,b}^1), \mathrm{u}_1\left(q_b^1\right)\Big) \in \mathcal{A}_\epsilon^{*(n)}$$

which is possible iff n is sufficiently large and

$$\Delta_1^1 > I\left(\hat{\mathrm{Y}}_1^1; \mathrm{Y}_1 | \mathrm{V}_2^1, \mathrm{W}_1^1, \mathrm{U}_1\right) \tag{18}$$

$$\Delta_1^2 > I\left(\hat{\mathrm{Y}}_1^2; \mathrm{Y}_1 | \mathrm{V}_2^1, \mathrm{W}_1^1, \mathrm{U}_1\right). \tag{19}$$

Relay level 2 does the following decoding steps:

- Using the channel output $\mathrm{y}_2(b)$ the node decodes $\mathrm{v}_1(s_{1,b}^1)$, $\mathrm{w}_1^1(z_{1,b}^1|s_{1,b}^1)$ and if $o_1(2) = 2$ also $\mathrm{w}_1^2(z_{1,b}^2|z_{1,b}^1, s_{1,b}^1)$ given $\mathrm{w}_{2,b}^1(z_{2,b}^1|\mathbf{s}_{2,b}^{[1;2]})$. This can be done almost error free iff n is sufficiently large and

$$R_1^1 < I\left(\mathrm{V}_1^1; \mathrm{Y}_2 | \mathrm{W}_2^1\right), \tag{20}$$

$$\hat{R}_1^1 < I\left(\mathrm{W}_1^1; \mathrm{Y}_2 | \mathrm{V}_1^1, \mathrm{W}_2^1\right) \tag{21}$$

$$\hat{R}_1^2 < I\left(\mathrm{W}_1^2; \mathrm{Y}_2 | \mathbf{W}_{[1;2]}^1\right), \text{if } o_1(2) = 2. \tag{22}$$

- Knowing $\mathrm{v}_1^1(s_{1,b}^1)$ the relay now decodes $\mathrm{u}_1(q_{b-1}^1)$ almost error free given $\mathrm{w}_1^{\phi_1(2)}(z_{1,b-1}^{\phi_1(2)}|z_{1,b-1}^{\phi_1(2)-1}, s_{1,b-1}^1)$ and $\mathrm{w}_2^1(z_{2,b-1}^1|\mathbf{s}_{2,b-1}^{[1;2]})$ (both are known from the last decoding step) iff n is sufficiently large and

$$R_s^1 < R_1^1 + I\left(\mathrm{U}_1; \mathrm{Y}_2 | \mathrm{W}_1^{\phi_1(2)}, \mathrm{W}_2^1\right). \tag{23}$$

- We know that each $\tilde{r}_{1,b-1}^1$ satisfying

$$\Big(\mathrm{y}_2(b-1), \hat{\mathrm{y}}_1^1\left(\tilde{r}_{1,b-1}^1|s_{2,b-1}^1, s_{1,b-1}^1, z_{1,b-1}^1, q_{b-1}^1\right), \mathrm{w}_1^1(z_{1,b-1}^1|s_{1,b-1}^1), \mathrm{w}_1^{\phi_1(2)}(z_{1,b-1}^{\phi_1(2)}|z_{1,b-1}^{\phi_1(2)-1}, s_{1,b-1}^1), \mathrm{v}_2^1(s_{2,b-1}^1), \mathrm{u}_1(q_{b-1}^1), \mathrm{w}_{2,b-1}^1(z_{2,b-1}^1|\mathbf{s}_{2,b-1}^{[1;2]})\Big) \in \mathcal{A}_\epsilon^{*(n)} \tag{24}$$

could be the correct index. Therefore, we define the set

$$\mathcal{L}_{2,r_{1,b-1}^1}\left(\mathrm{y}_2(b-1)\right) = \left\{\tilde{r}_{1,b-1}^1 : \tilde{r}_{1,b-1}^1 \text{satisfies (24)}\right\}$$

and select one index using $z_{1,b}^1$ iff

$$\exists \hat{r}_{1,b-1}^1 : \hat{r}_{1,b-1}^1 = \mathcal{Z}_1^1(z_{1,b}^1) \cap \mathcal{L}_{2,r_{1,b-1}^1}\left(\mathrm{y}_2(b-1)\right).$$

This decoding step is successful, i.e., $\hat{r}_{1,b-1}^1 = r_{1,b-1}^1$, iff n is sufficiently large and

$$\Delta_1^1 < \hat{R}_1^1 + \begin{cases} I(\mathrm{Y}_2\mathrm{W}_2^1; \hat{\mathrm{Y}}_1^1|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) & \text{if } \phi_1(2) = 1, \\ I(\mathrm{Y}_2\mathrm{W}_2^1\mathrm{W}_1^2; \hat{\mathrm{Y}}_1^1|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) & \text{otherwise.} \end{cases} \tag{25}$$

If $o_1(2) = 2$, the relay can decode in a similar way $r_{1,b-1}^2$ iff n is sufficiently large and

$$\Delta_1^2 < I(\mathrm{Y}_2\mathrm{W}_2^1\mathrm{W}_1^2; \hat{\mathrm{Y}}_1^2|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) + \hat{R}_1^1 + \hat{R}_1^2. \tag{26}$$

- Now the relay uses this quantization to decode q_{b-1}^2 iff n is sufficiently large and

$$R_s^2 < I\left(U_2; \mathrm{Y}_2, \hat{\mathrm{Y}}_1^{\phi_1(2)}|\mathbf{W}_{[1;2]}^1, \mathrm{W}_1^{\phi_1(2)}, \mathrm{V}_2^1, \mathrm{U}_1\right). \tag{27}$$

- In the same way as described for (18)-(19) the second level finds the index $r_{2,b-1}^1$ for quantization $\hat{\mathrm{y}}_2^1$ iff n is sufficiently large and

$$\Delta_2^1 > I\left(\mathrm{Y}_2; \hat{\mathrm{Y}}_2^1|\mathbf{W}_{[1;2]}^1, \mathrm{U}_2\right). \tag{28}$$

Finally, **the destination** does the following decoding steps

- At first the destination decodes $\mathrm{v}_2^1(s_{2,b}^1)$, $\mathrm{v}_2^2(s_{2,b}^2|s_{2,b}^1)$ and $\mathrm{w}_2^1(z_{2,b}^1|\mathbf{s}_{2,b}^{[1;2]})$ iff n is sufficiently large and

$$R_2^1 < I(\mathrm{V}_2^1; \mathrm{Y}_3), \tag{29}$$

$$R_2^2 < I(\mathrm{V}_2^2; \mathrm{Y}_3|\mathrm{V}_2^1), \tag{30}$$

$$\hat{R}_2^1 < I\left(\mathrm{W}_2^1; \mathrm{Y}_3|\mathrm{V}_2^2\right). \tag{31}$$

  Knowing $s_{2,b}^1$ the destination then decodes $\mathrm{v}_1^1(s_{1,b-1}^1)$ iff n is sufficiently large and

$$R_1^1 < I\left(\mathrm{V}_1^1; \mathrm{Y}_3|\mathrm{W}_2^1\right) + R_2^1. \tag{32}$$

- Using $s_{1,b-1}^1$ and the already in block $b-2$ decoded relay messages it can decode $\mathrm{u}_1(q_{b-2}^1)$ iff n is sufficiently large and

$$R_s^1 < I\left(\mathrm{U}_1; \mathrm{Y}_3|\mathrm{W}_1^{\phi_1(3)}, \mathrm{W}_2^1\right) + R_1^1. \tag{33}$$

- In the same way as shown in the description of (25) and (26) the destination at first determines $z_{1,b-1}^1$ and if $o_1(2) = d$ also $z_{1,b-1}^2$ iff n is sufficiently large and

$$\hat{R}_1^1 < I\left(\mathrm{W}_1^1; \mathrm{Y}_3|\mathrm{V}_1^1, \mathrm{W}_2^1\right) \tag{34}$$

$$\hat{R}_1^2 < I\left(\mathrm{W}_1^2; \mathrm{Y}_2|\mathbf{W}_{[1;2]}^1\right), \text{if } o_1(2) = d. \tag{35}$$

  Using these indices the destination determines the indices $r_{1,b-2}^k$, $k \in [1; \phi_1(d)]$, iff n is sufficiently large and

$$\Delta_1^1 < \hat{R}_1^1 + \begin{cases} I(\mathrm{Y}_3\mathrm{W}_2^1; \hat{\mathrm{Y}}_1^1|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) & \text{if } \phi_1(d) = 1, \\ I(\mathrm{Y}_3\mathrm{W}_2^1\mathrm{W}_1^2; \hat{\mathrm{Y}}_1^1|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) & \text{otherwise,} \end{cases} \tag{36}$$

  and if $o_1(2) = d$

$$\Delta_1^2 < I(\mathrm{Y}_3\mathrm{W}_2^1\mathrm{W}_1^2; \hat{\mathrm{Y}}_1^1|\mathrm{W}_1^1\mathrm{V}_2^1\mathrm{U}_1) + \hat{R}_1^1 + \hat{R}_1^2. \tag{37}$$

- Now the destination uses this quantization and the index $s_{2,b}^2$ to decode $\mathrm{u}_2(q_{b-2}^2|q_{b-2}^1)$ which is almost error free iff n is sufficiently large and

$$R_s^2 < I\left(U_2; \mathrm{Y}_3, \hat{\mathrm{Y}}_1^{\phi_1(3)}|\mathbf{W}_{[1;2]}^1, \mathrm{W}_1^{\phi_1(3)}, \mathrm{V}_2^1, \mathrm{U}_1\right) + R_2^2. \tag{38}$$

- As shown in the description of (25) and (26) the destination determines the index $r_{2,b-2}^1$ almost error free iff n is sufficiently large and

$$\Delta_2^1 < \hat{R}_2^1 + I(\mathrm{Y}_3, \hat{\mathrm{Y}}_1^1; \hat{\mathrm{Y}}_2^1|\mathrm{V}_2^1, \mathbf{W}_{[1;2]}^1, \mathbf{U}_{[1;2]}) \tag{39}$$

  if $\phi_1(3) = 1$ and otherwise

$$\Delta_2^1 < \hat{R}_2^1 + I(\mathrm{Y}_3, \hat{\mathrm{Y}}_1^2, \mathrm{W}_1^2; \hat{\mathrm{Y}}_2^1|\mathrm{V}_2^1, \mathbf{W}_{[1;2]}^1, \mathbf{U}_{[1;2]}). \tag{40}$$

- Finally the destination uses the quantizations of both relay levels to decode $\mathrm{u}_3\left(q_{b-2}^3|q_{b-2}^{[1;2]}\right)$ which is almost error free iff n is sufficiently large and

$$R_s^3 < I\left(U_3; \mathrm{Y}_3, \hat{\mathrm{Y}}_1^{\phi_1(3)}, \hat{\mathrm{Y}}_2^1|\mathbf{U}_{[1;2]}, \mathbf{W}_{[1;2]}^1, \mathrm{W}_1^{\phi_1(3)}, \mathrm{V}_2^1\right). \tag{41}$$

Using standard manipulations it is easy to show that (17)-(41) determine the achievable rates stated in Theorem 1.